# A Random Walk through Seasonal Adjustment:

# Noninvertible Moving Averages and Unit Root Tests

by

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ABSTRACT

This paper examines the distributions of (zero frequency) unit root test statistics for $I(1)$ processes in the presence of noninvertible moving average components. The analysis initially considers a noninvertible MA(1), for which the asymptotic distribution of the ADF test statistic under the unit root null hypothesis is shown to depend on the order of augmentation and can be shifted to either the right or the left, so that undersizing or oversizing problems may result. Although the distribution of the PP statistic depends on both the order of autocorrelation allowed and the weighting function used, it is always undersized with the Bartlett window. When extended to noninvertibility arising from X-11 seasonal adjustment of a random walk, the analytical features of the asymptotic distributions of these tests show corresponding characteristics as for the MA(1) case. These results are supported by a Monte Carlo analysis of the large sample distributions, and finite sample size properties of these unit root tests are also examined for a range of seasonal and nonseasonal $I(1)$ processes.

## 1. Introduction

The implications of seasonal adjustment have been studied by many authors since the pioneering works of Wallis (1974) and Sims (1974). Recent analyses include del Barrio Castro and Osborn (2004), Ericsson, Hendry and Tran (1994), Franses (1995, 1996), Ghysels (1990), Ghysels and Perron (1993, 1996), Ghysels and Liebermann (1996), Matas-Mir and Osborn (2004) and Otero and Smith (2002). Although these studies establish nontrivial consequences for seasonal adjustment in terms of shortrun properties, the general conclusion with respect to longrun properties is reassuring, with seasonal adjustment found to have no asymptotic impact on tests under the null hypothesis of (zero frequency) integration and cointegration; see, in particular, Ghysels and Perron (1993) and Ericsson *et al.* (1994).

These results are, however, open to question, since they rest on an invertibility assumption that is, in general, invalid for seasonally adjusted data. Indeed, the present paper shows that the existence of a noninvertible moving average unit root of -1 has nontrivial consequences for the asymptotic properties of zero frequency unit root tests. More specifically, depending on the order of augmentation adopted, the asymptotic distribution of the usual augmented Dickey-Fuller (Dickey and Fuller, 1979) [ADF] test statistic under the unit root null hypothesis can be shifted to either the right or the left, so that undersizing or oversizing may result. On the other hand, the Phillips-Perron (1988) [PP] statistic is undersized, irrespective of the order of autocorrelation allowed.

Due to the properties of the filter embedded in X-11 and its more recent development X-12-ARIMA, seasonal adjustment by these procedures may be expected to give rise to noninvertible moving average terms in the adjusted data (Maravall, 1993). However, if the usual unit root tests do not satisfactorily deal with noninvertible moving average components, then inferences (even asymptotically) about the presence of unit roots can be unreliable for seasonally adjusted data. We illustrate these effects analytically and through

Monte Carlo simulation, for both the ADF and PP tests. The analysis we undertake is related to that of Galbraith and Zinde-Walsh (1999), who examine the impact of moving average components on ADF tests. However, in contrast to their assumption of invertibility, we focus on the noninvertible moving average case and, more specifically, on the effect of seasonal adjustment. Also, although Ghysels and Perron (1993) examine the impact of seasonal adjustment on unit root tests, they assume invertibility.

The issue we study has not, to our knowledge, been considered in the literature. Maravall (1993) discusses the noninvertibility implication of seasonal adjustment, and hence recommends that unit root tests based on autoregressive augmentation should not be undertaken with seasonally adjusted data. However, he does not analyze the resulting asymptotic distributions. Although the Monte Carlo analyses of Ghysels (1990), Ghysels and Perron (1993) and Smith and Otero (2002) indicate size problems for univariate unit root or cointegration tests after seasonal adjustment, this is seen to be a finite sample issue. In contrast, we argue that the problem is more fundamental, since it affects the asymptotic distributions.

The paper is organised as follows. Section 2 contains some general discussion of seasonal adjustment and unit roots. Section 3 then analytically examines the ADF and PP (zero frequency) unit root tests in the presence of a noninvertible (seasonal) moving average root of -1. Section 4 generalizes the discussion to the case of seasonal adjustment, and contains both analytical and Monte Carlo results. Section 5 concludes.

## 2. Seasonal adjustment and moving average components

The regression

$$y_t = \rho\, y_{t-1} + u_t \qquad (1)$$

is the basis of all (zero frequency) unit root tests, with the relevant null hypothesis $\rho = 1$ or, equivalently, $\alpha = \rho - 1 = 0$. The disturbance innovations $u_t$ in (1) may exhibit temporal dependence and/or heteroskedasticity, with the limiting distribution of the normalized bias and *t*-ratio statistics for testing this null hypothesis given by Phillips (1987, Theorem 3.1). Our interest focuses on temporal dependence, with the typical assumption in unit root analyses (for example, Ghysels and Perron, 1993, Elliot, Rothenberg and Stock, 1996, Galbraith and Zinde-Walsh, 1999) being that the process for $u_t$ is stationary and invertible.

However, the invertibility of $u_t$ may be questioned when the series under analysis has been seasonally adjusted. Such procedures, including the widely-used X-11 or X-12 ARIMA program, typically assume the presence of nonstationary stochastic seasonality. More particularly, seasonal adjustment by X-11 can be represented as the application of a sequence of linear filters, with Laroque (1977) being the first to derive the implied filter coefficients in the quarterly case while Ghysels and Perron (1993) present the corresponding coefficients for monthly data. Although in many cases the use of X-11 has been replaced by X-12-ARIMA, the core features of X-11 seasonal adjustment remain essentially unchanged in this procedure (see Findley *et al*., 1998).

All filters routinely applied during the process of adjustment make implicit assumptions about the form of the process generating the unadjusted series. As shown by Burridge and Wallis (1984), the implied process for X-11 has one or two zero frequency unit roots and a full set of seasonal unit roots. These implied seasonal unit roots are a consequence of the moving annual summation operator[1] $S(L) = 1 + L + \ldots + L^{s-1}$, where $s$ is the frequency of observations per year (typically $s = 4$ or 12) and $L$ is the usual lag operator, which is embedded in X-11. In other words, for an unadjusted series, $y_t^u$, conventional X-11 seasonal adjustment assumes that the data generating process (DGP) is of the form

$$(1-L)^d S(L) y_t^u = w_t \qquad (2)$$

where $d = 1$ or 2 (with the best-fitting model implying $d = 2$) and $w_t$ is a moving average (MA) process (Burridge and Wallis, 1984). Approaches to seasonal adjustment based explicitly on unobserved components models also typically make this assumption; see, for example, Bell and Hillmer (1984) or Harvey (1989).

Despite the common use of seasonally adjusted data, empirical studies of the properties of seasonal time series find, in general, little evidence for the presence of the full set of seasonal unit roots implied by the autoregressive operator $S(L)$ in (2); see, among others Beaulieu and Miron (1993), Osborn (1990), or the discussion in Ghysels and Osborn (2001, pp.90-91). In other words, while economic series are typically integrated (containing at least one zero frequency unit root), they are not seasonally integrated. Therefore, application of seasonal adjustment based on an assumption of a DGP of the form (2) when the true DGP has no seasonal unit roots will induce the full set of (seasonal) unit roots implied by $S(L)$ in the MA component. Consequently, the stylized fact that macroeconomic time series are not seasonally integrated implies that the disturbance $u_t$ in the unit root test regression of (1), when $y_t$ is seasonally adjusted, may be anticipated to be a noninvertible moving average process[2].

As shown by Phillips (1987), the distribution of tests for $\rho = 1$ in (1) depends on unknown parameters related to the serial correlation of the innovations. The two widely used

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1 Ghysels and Osborn (1991, pp.96-98) discuss the sequence of filters applied. The filter $S(L)$ is applied in a centred form, so that its mid-point corresponds to a specific observation.

2 Maravall (1993) discusses the noninvertibility implications of seasonal adjustment and trend estimation in the context of an unobserved components model.

approaches proposed to deal with this problem are those of Phillips (1987) and Phillips and Perron [PP] (1988), which relies on a nonparametric correction for serial correlation, and the approach due to Dickey and Fuller (1979) [DF], which deals with serial correlation by augmenting the test regression (1) with lagged differences of $y_t$.

The seminal study of Schwert (1989) showed that unit root tests of the DF form, with autoregressive augmentation, are poorly sized in the presence of moving average components in (1). The analyses of Galbraith and Zinde-Walsh (1999) and Gonzalo and Pitarakis (1998) show why such distortions occur. In particular, these studies establish the dependence of the size distortions on the order of augmentation adopted, so that such distortions exist even asymptotically. Due to the dependence on the augmentation, this result is compatible with the fact that, in the presence of an invertible MA component, a valid test can be obtained using only autoregressive (AR) augmentation provided this augmentation is sufficiently large (Said and Dickey, 1984).

However, while Galbraith and Zinde-Walsh (1999) and Gonzalo and Pitarakis (1998) analytically examine the implications of MA components in (1), both assume these to be invertible. Nevertheless, Galbraith and Zinde-Walsh hint at the importance of this assumption, by noting that the size distortions in the DF test are particularly difficult to deal with in the presence of a near-noninvertible MA root.

In contrast to the DF test, which relies on an AR approximation to a MA, the Phillips-Perron (1988) approach uses observed residuals from (1) to mimic the autocorrelation properties of $u_t$, typically up to some maximum lag. Provided that the value employed for this maximum lag is at least as large as the order of the true MA, this approach is particularly attractive in the context of moving averages. However, practical applications of the PP test require the use of a weighting function to ensure nonnegativity of estimated variances and, as discussed below, this weighting function is crucial in the context of a noninvertible MA.

The present paper focuses on the unstudied issue of the impact of a noninvertible moving average component on the distributions of conventional unit root tests under the null hypothesis. This issue is important whether the ADF or PP test is applied, because of the impact of MA seasonal unit roots induced as an unrecognised side-effect of seasonal adjustment.

## 3. Noninvertible moving averages

For macroeconomic time series, intra-year observations are typically available at quarterly or monthly frequency ($s$ = 4 or 12). In each case the operator $S(L)$ of (2) contains the seasonal root -1 and in this section we focus our analysis on MA processes containing this root.

Therefore, consider the process

$$y_t = y_{t-1} + u_t \qquad t = 1, 2, \ldots, T \qquad (3)$$

where $u_t = \varepsilon_t + \varepsilon_{t-1}$ and $\varepsilon_t \sim iid(0, \sigma^2)$. The moving average unit root of -1 in (3) implies a zero in the spectral density of $\Delta y_t$ at a frequency of $\pi$ (that is, at the frequency corresponding to cycles of length two periods).

Throughout our theoretical analysis, we assume zero starting values ($y_0 = \varepsilon_0 = \varepsilon_{-1} = 0$ in (3)) and consider only test regressions without deterministic components. This is to keep the analysis as simple as possible in order to focus on the essential feature of our analysis, namely the consequences of noninvertible moving averages. Neither generalization to nonzero starting values or the introduction of deterministic components would alter the essential results.

3.1 No correction for autocorrelation

Consider first a DF test regression applied to (3) without augmentation, namely

$$\Delta y_t = \alpha y_{t-1} + u_t, \qquad t = 1, 2, \ldots, T \tag{4}$$

where, under the data generating process (DGP) $\alpha = 0$ and $u_t = \varepsilon_t + \varepsilon_{t-1}$. Application of OLS to (4) yields

$$\hat{\alpha} = \frac{\sum_{t=1}^{T} y_{t-1} u_t}{\sum_{t=1}^{T} y_{t-1}^2}.$$

The asymptotic distributions of the normalized bias and *t*-ratio are given in the following Proposition. (See the Appendix for proofs of all Propositions.)

**Proposition 1.** *Let* $y_t$ *follow* (3) *with* $u_t = \varepsilon_t + \varepsilon_{t-1}$ and $\varepsilon_t \sim iid(0, \sigma^2)$. *The asymptotic distribution of the normalized bias test statistic in* (4) *is then given by:*

$$T\hat{\alpha} \Rightarrow \frac{1}{2}\frac{[W(1)^2 - 1] + 0.5}{\int [W(r)]^2 dr}. \tag{5}$$

*and that for the t-ratio test statistic is:*

$$t_{\hat{\alpha}} \Rightarrow \frac{1}{\sqrt{2}}\frac{[W(1)^2 - 1] + 0.5}{\left\{\int [W(r)]^2 dr\right\}^{1/2}}. \tag{6}$$

Here, and throughout the paper, $\Rightarrow$ means convergence in distribution and $W(r)$ is standard Brownian motion.

The implication is that when no allowance is made for the autocorrelation inherent in (3), the distribution of the normalized bias $T\hat{\alpha}$ in (5) is asymptotically shifted to the right by the amount $0.25/[\int W(r)^2 dr]$, compared with the case of uncorrelated innovations. Also if we compare (6) with the usual Dickey-Fuller distribution for the *t*-ratio, namely

$$t_{\hat{\alpha}} \Rightarrow \frac{1}{2}\frac{[W(1)]^2 - 1}{\left\{\int [W(r)]^2 dr\right\}^{1/2}}, \tag{7}$$

the distribution in (6) is both shifted to the right through the addition of 0.5 to the numerator and is also increased by a factor of √2.

Consequently, the unit root hypothesis test is undersized (compared to the nominal test size), whether the normalized bias or the *t*-ratio form of the test is applied. As seen in Table 1 for the *t*-ratio, the undersizing is severe, with the null hypothesis being rejected only one tenth of the number of times indicated by a nominal size of 5 percent.

It is unsurprising that the DF test is undersized when the process has a positive noninvertible MA(1) component. The next two subsections turn to the more interesting issue of the effectiveness of AR augmentation and the PP approach in correcting the size.

### 3.2 Autoregressive augmentation

Now consider the usual ADF regression

$$\Delta y_t = \alpha y_{t-1} + \sum_{i=1}^{p} \phi_i \Delta y_{t-i} + v_t \tag{8}$$

where the DGP is again given by (3) with $u_t = \varepsilon_t + \varepsilon_{t-1}$. As discussed in detail in the Appendix, under the null hypothesis $\alpha = 0$ the autoregressive augmentation of (8) results in $v_t$ following an MA($p$+1) disturbance process, with coefficients

$$\theta_i^p = (-1)^{i+1}\left(\frac{1}{p+1}\right), \quad i = 1, ...., p+1 \tag{9}$$

Notice that, as a consequence of the noninvertibility of the MA process, these coefficients do not decline towards zero as $i$ increases. Indeed, as also shown in the Appendix, the AR approximation does not account for the noninvertible MA seasonal unit root unit root -1, since this root remains in the MA process with coefficients given in (9).

The consequences for the ADF test statistics are examined in Proposition 2, which provides the asymptotic distributions of the normalized bias and $t$-ratio tests for this case.

**Proposition 2.** *Let $y_t$ follow* (3) *with* $u_t = \varepsilon_t + \varepsilon_{t-1}$ and $\varepsilon_t \sim iid(0, \sigma^2)$. *The ADF normalized bias and t-ratio test statistic test statistics in regression* (8) *then satisfy:*

$$T\hat{\alpha} \Rightarrow \begin{cases} \dfrac{1}{4}\dfrac{\left[W(1)^2 - 1\right] - \dfrac{1}{p+1}}{\int W(r)^2 dr} & p \text{ odd} \\ \\ \dfrac{1}{4}\dfrac{\left(\dfrac{p+2}{p+1}\right)\left[W(1)^2 - 1\right] + \dfrac{1}{p+1}}{\int W(r)^2 dr} & p \text{ even} \end{cases} \tag{10}$$

*and*

$$t_{\hat{\alpha}} \Rightarrow \begin{cases} \dfrac{1}{2}\dfrac{\left[W(1)^2 - 1\right] - \dfrac{1}{p+1}}{\left[\int W(r)^2 dr\right]^{1/2}\left[\dfrac{p+2}{p+1}\right]^{1/2}} & p \text{ odd} \\ \\ \dfrac{1}{2}\dfrac{\left(\dfrac{p+2}{p+1}\right)\left[W(1)^2 - 1\right] + \dfrac{1}{p+1}}{\left[\int W(r)^2 dr\right]^{1/2}\left[\dfrac{p+2}{p+1}\right]^{1/2}} & p \text{ even} \end{cases} \tag{11}$$

As $p$ increases, the distributions (10) and (11) approach the DF distributions for the normalized bias and $t$-ratio statistics, respectively. This result applies despite the MA unit root that remains in (8), and indicates that a sufficiently high order of augmentation renders the DF distribution appropriate even in the presence of a noninvertible MA.

Nevertheless, for any finite and odd $p$, both distributions are shifted to the left, whereas they are shifted to the right for even $p$, compared with the DF distributions. There is also a scaling effect in both cases, and this is dependent on the order of augmentation.

Overall, we anticipate that the unit root test will be asymptotically oversized for *p* odd and undersized for *p* even.

The quantiles of the empirical distribution corresponding to (11) are shown in Table 1 for *p* = 0, 1, 2, 3, 4, 8, 12, 16, 20, 40, 100, 200 based 15,000 replications and sample size *T* = 4,000. The shift of the distribution to the left (right) for *p* odd (even) is evident, as is the consequent size problems (for nominal size 5 percent). While the distribution is approaching the DF distribution (shown in the top row) as *p* increases, nontrivial undersizing remains even when an augmentation of *p* = 24 is used, illustrating the inadequacy of even a high order autoregression to account for this first order noninvertible MA process.

3.3 PP approach

Phillips (1987) and Phillips-Perron (1988) propose correcting the normalized bias and *t*-ratio statistics to take account of serial correlation in (4) through the use of

$$Z(\hat{\alpha}) = T\hat{\alpha} - \frac{1}{2}\frac{\left(s_l^2 - s_u^2\right)}{T^{-2}\sum_{t=1}^{T} y_{t-1}^2}. \tag{12}$$

and

$$Z(t_{\hat{\alpha}}) = \left(\frac{s_l}{s_u}\right)t_{\hat{\alpha}} - \frac{1}{2}\frac{\left(s_l^2 - s_u^2\right)}{s_l\sqrt{T^{-2}\sum_{t=1}^{T} y_{t-1}^2}} \tag{13}$$

respectively, where

$$\begin{aligned} s_u^2 &= T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 \\ s_l^2 &= T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 + 2T^{-1}\sum_{i=1}^{p} w(i,p)\sum_{t=i+1}^{T}\hat{u}_t\hat{u}_{t-i} \end{aligned} \tag{14}$$

in which *p* is the truncation parameter, $\hat{u}_t$ (*t* = 1, …, *T*) are the residuals from an ordinary least squares estimation of (4) and *w*(*i, p*) is a weighting (or kernel) function used to ensure that the estimated longrun variance $s_l^2$ is nonnegative. Perhaps the most widely used weighting function in practice is the Bartlett window which has $w(i,p)=1-[i/(p+1)]$.

Proposition 3 obtains the asymptotic distributions for the PP statistics of (12) and (13) for the noninvertible MA(1) process of interest.

**Proposition 3.** *Let* $y_t$ *follow* (3) *with* $u_t = \varepsilon_t + \varepsilon_{t-1}$ and $\varepsilon_t \sim iid(0, \sigma^2)$. *Then the asymptotic distributions of the PP unit root test statistics of* (12) *and* (13) *are given by:*

$$Z(\hat{\alpha}) \Rightarrow \frac{1}{2}\frac{[W(1)^2-1]}{\int [W(r)]^2 dr} + \frac{0.25(1-w(1,p))}{4\int [w(r)]^2 dr} \quad (15)$$

$$Z(t_{\hat{\alpha}}) \Rightarrow \frac{1}{(2+2w(1,p))^{1/2}}\frac{[W(1)^2-1]}{\left[\int [W(r)]^2 dr\right]^{1/2}} + \frac{0.5(1-w(1,p))}{(2+2w(1,p))^{1/2}\left[\int [W(r)]^2 dr\right]^{1/2}}. \quad (16)$$

The weighting $w(1, p)$ applied to the first-order sample autocovariance in (14) enters the asymptotic distributions (15) and (16). Indeed, it is easy to see that the PP statistic distributions (15) and (16) are the usual asymptotic DF distributions only if $w(1, p) = 1$, which implies that no weighting is applied to this sample autocovariance. However, the Bartlett window uses $w(1,p)=p/(p+1)$, and in this case the distributions tend to the corresponding DF ones as $p \to \infty$. However, for finite $p$, the distributions are shifted to the right in relation to the DF case, with $Z(t_{\hat{\alpha}})$ also being subject to a scaling factor that is greater than unity. Indeed, it is easy to see from (16) that, for given $p$, the Bartlett window yields the asymptotic distribution

$$Z(t_{\hat{\alpha}}) \Rightarrow \frac{1}{2}\left[\frac{p+1}{p+0.5}\right]^{1/2}\left\{\frac{[W(1)^2-1]+0.5[p/(p+1)]}{\left[\int [W(r)]^2 dr\right]^{1/2}}\right\}. \quad (17)$$

Table 2 presents results for the distribution of $Z(t_{\hat{\alpha}})$, analogous to those for the DF $t$-statistic shown in Table 1. Although $p$ plays a different role in these two approaches, we again use $p$ = 1, 2, 3, 4, 8, 12, 16, 20, 40, 100, 200. In contrast to the ADF statistic, the statistic of (17) is always undersized, except for the far left-hand tail with relatively large $p$. However, for a nominal 5 percent significance level, the test is reasonably well sized for $p \geq 8$, since this provides relatively high weight to the first-order sample autocovariance in relation to the required $w(1, p) = 1$.

## 4. Seasonally adjusted random walk

We now turn to our case of principal interest, namely that of seasonal adjustment. To keep the analysis as simple as possible, while illustrating the implications of seasonal adjustment, assume that the true DGP for the unadjusted data series ($y_t^u$) is the simple random walk

$$y_t^u = y_{t-1}^u + \varepsilon_t, \qquad t = 1, 2, ..., T \tag{18}$$

where (again for simplicity) $\varepsilon_t = 0$ for $t \leq 0$. Indeed, the random walk is appropriate for analysis because it provides the only case of an $I(1)$ process where all the autocorrelation characteristics are induced by the adjustment filter. Therefore, studying this process allows us to focus on the impact of the filter.

We analyze the effect of X-11 seasonal adjustment using its default options, as widely used in practice. This adjustment can be approximated by a two sided symmetric linear filter[3]. Consequently, we write the process after adjustment (denoted $y_t^f$) as

[3] This applies to the central observations of the sample, where there are sufficient observations before and after the specific observation for the symmetric two-sided filter to be applied. Since we are concerned with the effects of the "typical" seasonal adjustment filter, we do not consider the effects of the asymmetric filters that are used in X-11 for observations at the beginning and end of the sample.

$$
\begin{aligned}
y_t^f &= y_{t-1}^f + u_t \\
u_t &= \sum_{i=k}^{-k} q_{|i|}\varepsilon_{t+i} = q(L)\varepsilon_t
\end{aligned}
\tag{19}
$$

where the coefficients $q_i$ are known (see Laroque, 1977, Ghysels and Perron, 1993). It should be noted that the nonzero weights (or coefficients) extend over a relatively long time span; for example, in the quarterly case $q_i \neq 0$ for $i = 0, 1, \ldots, 27$. Although the weights sum to unity, they are not all positive.

As discussed in Section 2, the X-11 seasonal adjustment filter applied to (18) results in the presence of seasonal unit roots in the MA of (19). Clearly, therefore, the filtered process retains the autoregressive unit root of (18), but is distorted through the complicated and noninvertible moving average introduced in the disturbances $u_t$. Using the Beveridge-Nelson (1981) decomposition, the filtered series can be written (see the Appendix) as:

$$
\begin{aligned}
y_t^f &= \sum_{s=1}^{t} u_s \\
&= \sum_{s=1}^{t} \varepsilon_s - \sum_{j=0}^{k-1}\left(\varepsilon_{t-j}\sum_{i=j+1}^{k} q_i\right) - \sum_{j=1}^{k}\left(\varepsilon_j \sum_{i=j}^{k} q_i\right) + \sum_{j=1}^{k}\left(\varepsilon_{t+j}\sum_{i=j}^{k} q_i\right)
\end{aligned}
\tag{20}
$$

when we assume that the two-sided symmetric filter is used for all observations[4] $t = 1, ..., T$. Expression (20) is useful in allowing us to obtain the distributions of the unit root test statistics.

In addition to the analytical analysis of the effect of seasonal adjustment on a random walk process in subsections 4.1 to 4.3, subsection 4.4 presents the results from a finite sample Monte Carlo study for a wide range of $I(1)$ processes.

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4 This mimics the situation where a researcher uses a historical sample of seasonally adjusted observations. Due to the two-sided filter, this implicitly assumes that observations to $t = T+k$ are available for $y_t$.

4.1 No correction for autocorrelation

The test regression for the DF test without augmentation applied to the filtered process (19) is

$$\Delta y_t^f = \alpha y_{t-1}^f + u_t . \quad (21)$$

As shown in following proposition, the asymptotic distributions of the unit root test statistics depend on the filter coefficients $q_i$ of (19).

**Proposition 4.** *For an unadjusted series following the random walk process of* (18)*, assume that the linear filter of* (19) *is applied. Then for test regression* (21), *the normalized bias has asymptotic distribution*

$$T\hat{\alpha} \Rightarrow \frac{1}{2}\frac{\left(\left[W(1)^2 - 1\right] + \left[1 - \sum_{j=-k}^{k} q_j^2\right]\right)}{\int W(r)^2\,dr}. \quad (22)$$

*while the asymptotic distribution for the t-ratio statistic is*

$$t_{\hat{\alpha}} \Rightarrow \frac{1}{2}\frac{\left(\left[W(1)^2 - 1\right] + \left[1 - \sum_{i=-k}^{k} q_i^2\right]\right)}{\sqrt{\int W(r)^2\,dr}\sqrt{\sum_{i=-k}^{k} q_i^2}}. \quad (23)$$

As for the DF regression without augmentation analyzed in Section 3, the distribution of the normalized bias in (22) is shifted to the right compared with the usual DF one. The numerator shift term $[1 - \Sigma q_j^2]$ is 0.174 and 0.214 for quarterly and monthly data, respectively. In the case of the distribution of the *t*-ratio, (23), when compared with the usual DF disribution of (7), both the numerator and denominator are affected by adjustment. The numerator shift is the same as in (22). The denominator scaling by the root of the sum of the squared filter weights is substantial, being 0.909 and 0.887 in the quarterly and monthly cases, respectively. Overall, we anticipate that the seasonally adjusted random walk process will result in undersized DF test statistics when no allowance is made for the autocorrelation in this process.

4.2 Autoregressive augmentation

Now consider the ADF regression (8) applied to the filtered series of (19). The DGP is again the random walk of (18). As for the noninvertible moving average of Section 3, autoregressive augmentation results in a moving average error process, with MA coefficient values that can be computed exactly. In terms of the disturbances of (18), this MA is two-sided and can be written as (see the Appendix)

$$\theta^p(L)\varepsilon_t = \theta^p_{-k}\varepsilon_{t+k} + \cdots + \theta^p_0\varepsilon_t + \cdots + \theta^p_k\varepsilon_{t-k} + \cdots + \theta^p_{k+p}\varepsilon_{t-k-p} \qquad (24)$$

where $k$ is the maximum lag of the filter in (19). Proposition 5 establishes how these MA coefficients affect the asymptotic distributions of the unit root tests.

**Proposition 5.** *For an unadjusted series following the random walk process of* (18)*, the ADF regression* (8) *applied to the filtered series of* (19) *has normalized bias and t-ratio test statistics that satisfy:*

$$T\hat{\alpha} \Rightarrow \frac{1}{2}\frac{\theta^p(1)\left[W(1)^2 - 1\right] + 2\left(\theta^p(1)\sum_{i=1}^{k} q_i - \sum_{i=1}^{k}\theta^p_{-i}\sum_{j=1}^{i} q_j + \sum_{i=1}^{k+1}\theta^p_i\sum_{j=0}^{i-1} q_j + \sum_{i=k+2}^{k+p}\theta^p_i\sum_{j=0}^{k} q_j\right)}{\int W(r)^2\,dr} \qquad (25)$$

*and*

$$t_{\hat{\alpha}} \Rightarrow \frac{1}{2}\frac{\theta^p(1)\left[W(1)^2 - 1\right] + 2\left(\theta^p(1)\sum_{i=1}^{k} q_i - \sum_{i=1}^{k}\theta^p_{-i}\sum_{j=1}^{i} q_j + \sum_{i=1}^{k+1}\theta^p_i\sum_{j=0}^{i-1} q_j + \sum_{i=k+2}^{k+p}\theta^p_i\sum_{j=0}^{k} q_j\right)}{\sqrt{\int W(r)^2\,dr}\sqrt{\left(\left(\theta^p_{-k}\right)^2 + \cdots + \left(\theta^p_0\right)^2 + \cdots + \left(\theta^p_k\right)^2 + \cdots + \left(\theta^p_{k+p}\right)^2\right)}}. \qquad (26)$$

*respectively, where* $\theta^p_i$ *are defined in* (24) *and* $\theta^p(1) = \sum_{i=-k}^{k+p}\theta^p_i$ .

The normalized bias in (25) has both scaling and shift effects, compared with the DF distribution. Table 3 computes these effects for augmentation values $p = 0, 1, 2, 3, 4, 8, \ldots, 20, 40, 100, 200$ for quarterly data. As in the simple noninvertible case of Section 3, the shift effect that applies for the ADF test in both (25) and (26), compared with the corresponding DF distribution, can be either to the left or to the right, depending on the order $p$ selected. Indeed, the shift is more marked for $p = 3$ than for lower orders of augmentation and in this case the shift is to the left. However, the shift is to the right for values of $p$ that are multiples of 4. When considering the normalized bias in (25) the numerator scaling effect of $\theta^p(1)$ also applies. Although this is relatively unimportant for $p = 4$, the decline in $\theta^p(1)$ away from unity as $p$ increases implies that the normalized bias statistic may not approach the DF distribution as $p$ increases. Although they do not derive the analytical distribution as in (25), Ghysels and Perron (1993) also note that the asymptotic distribution of the normalized bias statistic is affected by seasonal adjustment.

For the more commonly used $t$-ratio test, a denominator scaling also applies in (26) compared with the DF distribution of (7). The final two columns of Table 3 indicate that both the ratio of the two scaling effects and the shift tend to decline (toward unity and zero, respectively) as $p$ increases. However, this is not monotonic; indeed, the scaling ratio and the scaled shift are larger with $p = 20$ than the corresponding values when no augmentation is used. For intermediate values of $p$, and considering values that are multiples of four, the scaling and shift effects will lead to even larger distortions compared with the DF distribution than for no augmentation. Therefore, although these effects are relatively unimportant for $p = 100$, the results imply that very large orders of augmentation (and well beyond those used in empirical studies, even with large samples) are required to render the asymptotic distribution of the ADF $t$-statistic in the seasonally adjusted random walk close to that of the DF distribution.

To provide more detail of these effects, Table 4 reports the quantiles of the empirical approximation to the asymptotic distribution of (26). A random walk without filtering (requiring no augmentation) and filtered by the X-11 linear approximation[5] (with augmentation orders as in Table 3) are considered. These results support the analytical ones, showing that the distribution can be shifted to the left or right, depending on the order of augmentation. Using a nominal significance level of 5 percent, the presence of over/undersizing depends on the sign of the numerator shift term. Oversizing is evident particularly when $p = 3$, for which the entire distribution in (26) is substantially shifted to the left compared with the DF distribution.

It is evident from Table 4 that increasing the order of augmentation does not necessarily lead to an improved approximation to the DF distribution, since relatively little distortion is obtained with the inclusion of only one augmentation lag, although the test is under-sized (at a nominal 5 percent level) even in this case. Due to the scaling effect in (26), shown in Table 3, the size distortion is particularly marked when $p$ is a multiple of 4. For these values, the shift is always positive and the test continues to be undersized. Indeed, the distribution in Table 4 is largely unchanged for $p = 4, 8, 12$ and is shifted further to the right in these cases than when the DF regression without augmentation is used. As discussed above, the distribution with $p = 20$ suffers greater distortion than $p = 0$, with this being particularly notable for the left-hand tail. Although reduced with $p = 40$, the distortion remains substantial, with these results again emphasizing the extremely high orders of augmentation required asymptotically to approximate the complicated and noninvertible moving average characteristics induced by seasonal adjustment.

[5] In order to apply the two-sided quarterly filter, 50 additional observations are generated at the beginning and at the end of the sample when filtered data are used, but only the central observations are used in the computations.

4.3 Phillips-Perron approach

Since, after seasonal adjustment, $u_t$ = $\sum_{i=k}^{-k} q_{|i|}\varepsilon_{t+i}$ in (19), the variance and autocovariances of $u_t$ are given by

$$\gamma_0 = E\left[u_t^2\right] = \sigma^2 \sum_{i=-k}^{k} q_i^2$$

$$\gamma_s = E\left[u_t u_{t-j}\right] = \sigma^2 \sum_{i=-k}^{k-s} q_i q_{i-s} \ .$$

The PP approach aims to account for these autocovariances nonparametrically, through (12)/(13), with Proposition 6 giving the resulting asymptotic distributions for the random walk DGP.

**Proposition 6.** *For an unadjusted series following the random walk process of* (18)*, the PP test statistics of* (12) *and* (13)*, applied to the seasonally adjusted series of* (19), *have asymptotic distributions*

$$Z(\hat{\alpha}) \Rightarrow \frac{1}{2}\frac{\left[W(1)^2 - 1\right] + \left[1 - \sum_{j=-k}^{k} q_j^2\right] - \left(2\sum_{i=1}^{p} w(i,p)\sum_{j=-k}^{k-i} q_j q_{j-i}\right)}{\int W(r)^2\, dr} \tag{27}$$

$$Z(t_{\hat{\alpha}}) \Rightarrow \frac{1}{2}\frac{[W(1)^2 - 1]}{\sqrt{\int W(r)^2\, dr}\sqrt{\sum_{i=-k}^{k} q_i^2 + 2\sum_{i=1}^{p} w(i,p)\sum_{j=-k}^{k-i} q_j q_{j-i}}}$$

$$+ \quad \frac{1}{2}\frac{\left[1 - \sum_{i=-k}^{k} q_i^2\right] - \left(2\sum_{i=1}^{p} w(i,p)\sum_{j=-k}^{k-i} q_j q_{j-i}\right)}{\sqrt{\int W(r)^2\, dr}\sqrt{\sum_{i=-k}^{k} q_i^2 + 2\sum_{i=1}^{p} w(i,p)\sum_{j=-k}^{k-i} q_j q_{j-i}}} \tag{28}$$

It is evident from (27) that the seasonal adjustment filter induces a shift term in $Z(\hat{\alpha})$, while both shift and scale effects are present in $Z(t_{\hat{\alpha}})$. Further, the weighting function appears in these expressions, and hence plays a role in the asymptotic distributions. Table 5 computes these shift and scale terms for selected values of $p$ in the case of the Bartlett window. The shift effect is especially marked for $p$ = 1 and 2, but is relatively modest for $p \geq 8$. Since the denominator scaling factor is also close to unity for such values of $p$, it is anticipated that $p \geq 8$ might be sufficient to account for most of the moving average effects, including the noninvertible roots, induced by seasonal adjustment.

To investigate further, Table 6 presents the simulated asymptotic distribution for $Z(t_{\hat{\alpha}})$, when the PP statistic applied to a seasonally adjusted random walk and the Bartlett window is employed. Although the distribution is always shifted to the right, as anticipated the shift is largely invariant to $p$, provided $p \geq 8$. However, the effect is not monotonic, and $p = 3$ also provides a good approximation to the DF $t$-ratio distribution. Notice that, although the shift terms for $p$ = 4, 8 in Table 5 are negative, the scale factors in these cases are greater than unity, and the test is always undersized (at the nominal 5 percent level) in Table 6.

It should also be noted that although the PP test applied to a seasonally adjusted random walk in Table 6 has better size properties (at the nominal 5 percent level), in general, than the ADF test in Table 4, the PP test remains undersized by around 10 percent even if sample autocovariances to order 200 are considered.

## 4.4 Finite sample Monte Carlo analysis

The above analysis establishes the potentially nontrivial asymptotic effects of seasonal adjustment on unit root tests. Therefore, we next investigate the effects for finite samples and a wider range of $I(1)$ processes.

Tables 7 and 8 show the empirical size obtained in a Monte Carlo analysis for a sample of $T$ = 200 observations, using the ADF $t$-ratio test and the PP $Z(t_{\hat{\alpha}})$ statistics, respectively, computed for unfiltered and filtered data when the nominal size is 5 percent. In each case we use $p$ = 0, 1, 2, 3, 4, 8, 12, where 12 represents the maximum order that might be used by a practitioner for a quarterly sample of this size[6].

The DGP is that of Ghysels and Perron (1993), with the unfiltered data generated from an unobserved components $I$(1) process as

$$\begin{aligned} y_t &= y_t^{ns} + y_t^s \\ y_t^{ns} &= y_{t-1}^{ns} + \varepsilon_t^{ns} + \theta\varepsilon_{t-1}^{ns} \\ y_t^s &= \phi_s y_{t-4}^s + \varepsilon_t^s + \theta_s\varepsilon_{t-4}^s \end{aligned} \qquad (29)$$

where both $\varepsilon$ in (29) are mutually uncorrelated independent standard Normal variables. The results shown are for regressions including an intercept[7]. In addition to the seasonal DGPs considered by Ghysels and Perron, we also examine three cases (including the random walk) where the DGP contains no seasonality[8], and hence $y_t^s = 0$, in order to allow comparison with the theoretical analysis above.

Note, first, that the finite sample properties for the test size of the seasonally adjusted random walk in Tables 7 and 8, shown as the first DGP, are very similar to the asymptotic properties of the ADF and PP tests Tables 4 and 6, respectively. Thus, for example, the oversizing of the ADF test with $p$ = 3 applied to adjusted data in Table 4 is not simply an asymptotic property and is repeated in Table 7, while the size is better when the test is not corrected for autocorrelation compared with the cases where an ADF test with orders of augmentation $p$ = 4, 8, 12 are used. Similarly, the PP test (using the Bartlett window) is

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6 Of course, the results are identical when $p$ = 0, and hence these are reported only in Table 7.

7 Results were also computed without an intercept, with both sets also computed for a sample of size $T$ = 400. These results were qualitatively very similar to those shown.

8 While it might be considered unrealistic to apply seasonal adjustment in such cases, it should be borne in mind that (29) allows only stochastic seasonality. Since deterministic seasonality is annihilated by seasonal adjustment, with the same effect on the stochastic properties as in the nonseasonal case, we anticipate that the addition of deterministic seasonality would not change the pattern of results we obtain.

always undersized for this DGP after seasonal adjustment in Table 8, as in Table 6, irrespective of the value of $p$ chosen. In almost all cases, the empirical size for this DGP is better using unadjusted data than adjusted series, whichever test is applied.

This last comment also applies to the ADF test applied to a nonseasonal MA with coefficient 0.5 in Table 7, but adjustment has little effect on the size of the PP test for this case in Table 8. Whether adjusted or not, the PP test is badly oversized when the MA coefficient is negative, whereas AR augmentation performs quite well for $p \geq 4$.

For the unfiltered data and for all DGPs of Table 7, reasonable empirical size are generally obtained for the ADF test applied with $p$ = 8, which is sufficient to account for (most of) the seasonal autocorrelation. However, for DGPs with a nonzero seasonal moving coefficient $\theta_s$, $p$ = 12 is sometimes required; see, for example, the combinations with $\theta$ = 0.5, $\theta_s$ = 0.5. In contrast, the PP test applied to unadjusted data in Table 8 is always substantially oversized when the DGP contains a seasonal component. This is due to the Bartlett window not allowing for seasonality, and giving less weight to low order nonseasonal lags than to seasonal ones.

Now we turn to a discussion of the empirical size of the ADF test for a seasonal time series after application of the seasonal adjustment filter. The unit root test has good size properties for some of these cases in Table 7, even without augmentation. This occurs when a strong positive seasonal autoregressive coefficient ($\phi_s$ = 0.5, 0.8, 0.9) combines with a positive nonseasonal MA coefficient ($\theta$ = 0.5), when the true DGP has similar empirical properties as the DGP for which X-11 is the optimal filter. Otherwise, the test is oversized for seasonal DGPs when the test regression is not augmented. It is notable that augmentation with $p$ = 4 generally improves the size, but higher orders quite often lead to a deterioration in size. This is in line with Table 2, where the distortionary effects of adjustment on the scaling ratio and scaled shift compared with the DF distribution are greater with $p = 8$ and 12 than with $p = 4$.

It is also notable that (with the exception of cases with negative $\theta$) reasonable empirical sizes are generally obtained for filtered data in Table 7 with $p = 1$ and $p = 2$. This appears to result from a combination of the relatively small distortion induced by seasonal adjustment with these augmentation orders (see Tables 2 and 3) and the empirical proximity of DGPs with positive nonseasonal MA and strong positive seasonal AR coefficients to the DGP implicitly assumed by X-11.

The PP tests applied to filtered data, Table 8, does not perform well for a purely seasonal process, and is always oversized when $\theta = 0$. Although seasonal adjustment reduces this oversizing, it nevertheless remains substantial. However, its performance is even worse in the presence of a negative MA(1) coefficient, where the oversizing is severe irrespective of the value of $p$ employed and whether the data are unadjusted or seasonally adjusted.

The only set of seasonal time series in Table 8 for which the PP test has approximately the correct size after seasonal adjustment are those where $\phi_s$ = 0.5, 0.8, 0.9 combines the positive nonseasonal MA with $\theta = 0.5$, and hence (as mentioned above) the true DGP has similar empirical properties to the DGP for which X-11 is the optimal filter. In this case, seasonal adjustment is successful in removing the seasonal component and the PP test performs well.

## 5. Concluding remarks

Our analysis has shown that large size distortions can result in the distributions of both (Augmented) Dickey-Fuller and Phillips-Perron unit root test statistics when these are applied to processes containing noninvertible seasonal moving average unit roots. Indeed, we believe that our analytical results are the first to be derived for these tests in the presence of such noninvertible moving averages.

For the case of a $I(1)$ process with such a moving average root, we show that autoregressive augmentation of any order does not remove this unit root and we obtain exact analytical expressions for the asymptotic DF distributions. The corresponding analysis for the PP tests emphasizes the role of the weighting function. Our analysis is extended to the case of a seasonally adjusted random walk, which contains the full set of seasonal moving average roots. Here very high orders of autoregressive augmentation are required to approximate the null DF distribution, whereas the PP test (although undersized) performs reasonably well when 8 or more sample autocovariances are considered.

In common with the results of Galbraith and Zinde-Walsh (1999) and Gonzalo and Pitarakis (1998), who study unit root tests in the presence of invertible moving average processes, we find that the asymptotic distribution of ADF statistics depend on the order of augmentation adopted. However, a surprising, and important, finding of our analysis is that increasing the order of augmentation does not necessarily lead to an asymptotic distribution for the ADF test that more closely approximates the corresponding DF one. Indeed, asymptotically, applying the ADF test with 20 lags to a quarterly seasonally adjusted random walk data results in worse distortion than no augmentation at all. Further, since the DF test statistics can be either under or oversized, depending on the order of augmentation, it is difficult for the applied worker to make any informal allowance for the distortions than may apply. Nevertheless, the use of augmentation orders that are multiples of four with quarterly data will typically result in undersizing. In contrast, the PP test is always undersized.

Despite the relatively accurate size for the PP test for a seasonally adjusted random walk, this does not carry over when the DGP is itself a seasonal series. Indeed, our results imply that the PP test is badly oversized when applied to a seasonal time series after seasonal adjustment, unless the DGP has characteristics (specifically, a strong positive seasonal autoregressive component and a positive nonseasonal moving average) that approximate the properties of the DGP for which X-11 seasonal adjustment is optimal.

Because of the complicated effects of adjustment, our recommendation is that unit root analysis should be applied to the seasonally unadjusted series. If an ADF approach is adopted, this should be combined with diagnostic testing that an appropriate order of augmentation is used. Because the weighting functions used in conjunction with the PP test typically allocate smaller weights to sample autocovariances at longer lags, this approach does not account for strong seasonal autocovariances and hence is not to be recommended for application to seasonal time series.

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# Appendix

**Proof of Proposition 1**

Phillips (1987) shows that the distribution of the normalized bias for any moving average process disturbance process $u_t = \theta(L)\varepsilon_t$, with $\sigma^2 = E(\varepsilon_t^2)$, is:

$$T\hat{\alpha} \Rightarrow \frac{1}{2}\frac{\{[W(1)]^2 - 1\} + \left\{1 - \gamma_0/\lambda^2\right\}}{\int [W(r)]^2\,dr} \tag{A.1}$$

and that for the $t$-ratio test statistic is:

$$t_{\hat{\alpha}} \Rightarrow \frac{\lambda}{2\sqrt{\gamma_0}}\frac{\left\{[W(1)]^2 - \gamma_0/\lambda^2\right\}}{\{\int [W(r)]^2\,dr\}^{1/2}}. \tag{A.2}$$

where $W(r)$ is standard Brownian motion, $\gamma_0 = E(u_t^2)$ and $\lambda^2 = [\theta(1)]^2\sigma^2$. For the noninvertible moving average $u_t = \varepsilon_t + \varepsilon_{t-1}$ of interest, $\gamma_0 = 2\sigma^2$ and $\lambda^2 = 4\sigma^2$, so that (A.1) and (A.2) become (5) and (6) respectively.

■

With augmentation and under the (true) null hypothesis in (8) of $\alpha = 0$, the corresponding "pseudo-true" process can be written

$$\Delta y_t = \sum_{i=1}^{p}\phi_i^p \Delta y_{t-i} + e_t^p. \tag{A.3}$$

OLS estimation of the vector $\phi^p = (\phi_1^p, \ldots, \phi_p^p)'$ in (A.3) yields

$$\hat{\phi}^p \to \phi^p = \Gamma^{-1}\gamma \tag{A.4}$$

with $\Gamma$ being the $(p \times p)$ covariance matrix for $\Delta y_t$ with $(i, j)$th element $E(\Delta y_{t-i}\Delta y_{t-j})$, and $\gamma$ is the $(p \times 1)$ vector of autocovariances with $j$th element $E(\Delta y_t \Delta y_{t-j})$ for $j = 1, \ldots, p$. When $u_t = \varepsilon_t + \varepsilon_{t-1}$, these take the simple form

$$\Gamma = \sigma^2\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \gamma = \sigma^2\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{A.5}$$

Note that the elements of $\phi^p$ depend on the autoregressive augmentation order $p$ selected. Specifically, using results for the inverse of the covariance matrix for an MA(1) process (Shaman, 1969; Galbraith and Galbraith, 1974), it follows for our case that

$$\gamma^{ij} = (-1)^{j-i}\frac{(i)\times(p-j+1)}{(p+1)\sigma^2} \qquad j \geq i \tag{A.6}$$

where $\gamma^{ij}$ is the $(i, j)$th element of $\Gamma^{-1}$. Therefore, using (A.5) and (A.6),

$$\phi_i^p = (-1)^{i+1}\left(\frac{p-i+1}{p+1}\right). \tag{A.7}$$

The "disturbance" series $e_t^p$ in (A.3) is a MA($p$+1) process, which is easy to see since

$$\begin{aligned} e_t^p &= \Delta y_t - \sum_{i=1}^{p}\phi_i^p \Delta y_{t-i} = (\varepsilon_t + \varepsilon_{t-1}) - \sum_{i=1}^{p}\phi_i^p(\varepsilon_{t-i}+\varepsilon_{t-i-1}) \\ &= \varepsilon_t + (1-\phi_1^p)\varepsilon_{t-1} - (\phi_1^p + \phi_2^p)\varepsilon_{t-2} - \ldots - (\phi_{p-1}^p + \phi_p^p)\varepsilon_{t-p} - \phi_p^p \varepsilon_{t-p-1}. \\ &= \varepsilon_t + \sum_{i=1}^{p+1}\theta_i^p \varepsilon_{t-i}. \end{aligned} \tag{A.8}$$

Using (A.7) and (A.8), yields the MA coefficients given in (9).

Therefore, the AR($p$) approximation in (A.3) to the DGP $\Delta y_t = \varepsilon_t + \varepsilon_{t-1}$ leads to an MA($p$+1) disturbance in the ADF regression, with the AR and MA coefficients given by (A.7) and (9) respectively. To see that the noninvertible MA seasonal unit root unit root -1 remains in the MA process with coefficients in (9), note that

$$\theta^p(-1) = 1 + \sum_{i=1}^{p+1}(-1)^i\theta_i^p = 1 + \sum_{i=1}^{p+1}(-1)^{2i+1}\left(\frac{1}{p+1}\right) = 1 - \sum_{i=1}^{p+1}\frac{1}{p+1} = 0 \tag{A.9}$$

and hence -1 is a root of this MA process.

**Proof of Proposition 2**

OLS estimation of (8) yields $\hat{\alpha} \to 0$ (Phillips, 1987) and, using (A.3),

$$\begin{bmatrix} \hat{\alpha}-0 \\ \hat{\phi}_1 - \phi_1^p \\ \hat{\phi}_2 - \phi_2^p \\ \vdots \\ \hat{\phi}_p - \phi_p^p \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{T} y_{t-1}^2 & \sum_{t=1}^{T} y_{t-1}\Delta y_{t-1} & \sum_{t=1}^{T} y_{t-1}\Delta y_{t-2} & \cdots & \sum_{t=1}^{T} y_{t-1}\Delta y_{t-p} \\ \sum_{t=1}^{T} y_{t-1}\Delta y_{t-1} & \sum_{t=1}^{T} \Delta y_{t-1}^2 & \sum_{t=1}^{T} \Delta y_{t-1}\Delta y_{t-2} & \cdots & \sum_{t=1}^{T} \Delta y_{t-1}\Delta y_{t-p} \\ \sum_{t=1}^{T} y_{t-1}\Delta y_{t-2} & \sum_{t=1}^{T} \Delta y_{t-1}\Delta y_{t-2} & \sum_{t=1}^{T} \Delta y_{t-2}^2 & & \sum_{t=1}^{T} \Delta y_{t-2}\Delta y_{t-p} \\ \vdots & \vdots & & \ddots & \vdots \\ \sum_{t=1}^{T} y_{t-1}\Delta y_{t-p} & \sum_{t=1}^{T} \Delta y_{t-1}\Delta y_{t-p} & \sum_{t=1}^{T} \Delta y_{t-2}\Delta y_{t-p} & \cdots & \sum_{t=1}^{T} \Delta y_{t-p}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^{T} y_{t-1}e_t^p \\ \sum_{t=1}^{T} \Delta y_{t-1}e_t^p \\ \sum_{t=1}^{T} \Delta y_{t-2}e_t^p \\ \vdots \\ \sum_{t=1}^{T} y_{t-p}e_t^p \end{bmatrix}$$

For the same reasons as when the true DGP is of the autoregressive form (see, for example, Hamilton, 1994, pp.516-527), different rates of convergence apply to the coefficients of the nonstationary and stationary variables in (8), leading us to consider

$$\begin{bmatrix} T\hat{\alpha} \\ T^{1/2}(\hat{\phi}-\phi^p) \end{bmatrix} = \begin{bmatrix} T^{-2}\sum_{t=1}^{T} y_{t-1}^2 & h' \\ h & \hat{\Gamma} \end{bmatrix}^{-1} \begin{bmatrix} T^{-1}\sum_{t=1}^{T} y_{t-1}e_t^p \\ g \end{bmatrix} \tag{A.10}$$

where the ($p$ × 1) vector $h$ has $j$th element $T^{3/2}\Sigma y_{t-1}\Delta y_{t-j}$ and the ($p$ × 1) vector $g$ has $j$th element $T^{1/2}\Sigma\, e_t^p \Delta y_{t-1}$.

In (A.10),

$$\begin{bmatrix} T^{-2}\sum_{t=1}^{T} y_{t-1}^2 & h' \\ h & \hat{\Gamma} \end{bmatrix} \rightarrow \begin{bmatrix} \lambda^2 \int [W(r)]^2 dr & 0 \\ 0 & \Gamma \end{bmatrix} = \begin{bmatrix} 4\sigma^2 \int [W(r)]^2 dr & 0 \\ 0 & \Gamma \end{bmatrix} \quad \text{(A.11)}$$

while

$$T^{-1}\sum_{t=1}^{T} y_{t-1} e_t^p = T^{-1}\sum_{t=1}^{T} y_{t-1}(\varepsilon_t + \theta_1^p \varepsilon_{t-1} + ... + \theta_{p+1}^p \varepsilon_{t-p-1})$$
$$= T^{-1}\sum_{t=1}^{T} \{ y_{t-1}\varepsilon_t + \theta_1^p y_{t-2}\varepsilon_{t-1} + ... + \theta_{p+1}^p y_{t-p-2}\varepsilon_{t-p-1}$$
$$+ \theta_1^p (y_{t-1} - y_{t-2})\varepsilon_{t-1} + ... + \theta_{p+1}^p (y_{t-1} - y_{t-p-2})\varepsilon_{t-p-1} \}$$

Standard results (see, for example, Hamilton, 1994, pp.505.506) imply that

$$T^{-1}\sum_{t=1}^{T} \varepsilon_t y_{t-1} \Rightarrow \frac{1}{2}\lambda\sigma[W(1)^2 - 1] = \sigma^2[W(1)^2 - 1]. \quad \text{(A.12)}$$

Further

$$T^{-1}\sum_{t=1}^{T} \varepsilon_{t-i}(y_{t-1} - y_{t-i}) = T^{-1}\sum_{t=1}^{T} \varepsilon_{t-i}[\sum_{j=1}^{i}(\varepsilon_{t-j} + \varepsilon_{t-j-1})] \rightarrow \begin{cases} \sigma^2 & i = 1 \\ 2\sigma^2 & i > 1 \end{cases}$$

and hence

$$T^{-1}\sum_{t=1}^{T} y_{t-1} e_t^p \Rightarrow \sigma^2 \theta^p(1)\left[W(1)^2 - 1\right] + \sigma^2\left(\theta_1^p + 2\sum_{i=2}^{p+1}\theta_i^p\right) \quad \text{(A.13)}$$

where $\theta^p(1) = 1 + \theta_1^p + ... + \theta_{p+1}^p$. Therefore,

$$T\hat{\alpha} \Rightarrow \frac{1}{4}\frac{\theta^p(1)[W(1)^2 - 1] + \left(\theta_1^p + 2\sum_{i=2}^{p+1}\theta_i^p\right)}{\int W(r)^2 dr}, \quad \text{(A.14)}$$

which collapses to (A.1) when $p = 0$.

Now, from (9),

$$\theta^p(1) = 1 + \sum_{i=1}^{p+1}\theta_i^p = \begin{cases} 1 & p \ odd \\ \dfrac{p+2}{p+1} & p \ even \end{cases} \quad \text{(A.15)}$$

and

$$\theta_1^p + 2\sum_{i=2}^{p+1}\theta_i^p = \begin{cases} -\dfrac{1}{p+1} & p \ odd \\ \dfrac{1}{p+1} & p \ even \end{cases} \quad \text{(A.16)}$$

then using (A.14), (A.15) and (A.16), it is straightforward to obtain (10).

For the *t*-ratio test we have that:

$$t_{\hat{\alpha}} = T\hat{\alpha} \times \frac{\sqrt{T^{-2}\sum_{t=1}^{T} y_{t-1}^2}}{\sqrt{T^{-1}\sum_{t=1}^{T}\left(\Delta y_t - \hat{\alpha} y_{t-1} - \sum_{i=1}^{p}\hat{\phi}_i^p y_{t-1}\right)^2}}$$

Knowing that

$$T^{-1}\sum_{t=1}^{T}\left(\Delta y_t - \hat{\alpha} y_{t-1} - \sum_{i=1}^{p}\hat{\phi}_i^p y_{t-1}\right)^2 \Rightarrow Var(e_t^p) = \sigma^2 \sum_{i=0}^{p+1}(\theta_i^p)^2 \qquad \text{(A.17)}$$

where, using (9),

$$\sqrt{\sum_{i=0}^{p+1}(\theta_i^p)^2} = \sqrt{1+\frac{p+1}{(p+1)^2}} = \sqrt{\frac{p+2}{p+1}}\,.$$

This, together with (A.11) and (10), yields the distribution for the *t*-ratio statistic in (11).

■

**Proof of Proposition 3**

As $u_t = \varepsilon_t + \varepsilon_{t-1}$, the autocovariances of $u_t$ are $\gamma_0 = E[u_t^2] = 2\sigma^2$, $\gamma_1 = E[u_t u_{t-1}] = \sigma^2$ and $\gamma_j = E[u_t u_{t-j}] = 0$ for $j>1$. Therefore, from (14) and under the null hypothesis $\alpha = 0$,

$$\begin{aligned} s_u^2 &= T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 \to \gamma_0 = 2\sigma^2 \\ s_l^2 &= T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 + 2T^{-1}\sum_{i=1}^{p} w(i,p)\sum_{t=i+1}^{T}\hat{u}_t\hat{u}_{t-i} \qquad \text{(A.18)} \\ &\to \gamma_0 + 2w(1,p)\gamma_1 = 2\sigma^2 + 2w(1,p)\sigma^2 \end{aligned}$$

where $\to$ indicates convergence in probability. Then (15) and (16) are easily obtained by substituting (5) and (6), together with (A.18) into (12) and (13), and also using $T^{-2}\sum_{t=1}^{T} y_{t-1}^2 \Rightarrow \lambda^2 \int [W(r)]^2 dr = 4\sigma^2 \int [W(r)]^2 dr$.

■

**Proof of (20)**

The filtered random walk of (19) is given by

$$
\begin{aligned}
y_t^f &= \sum_{s=1}^{t} u_s \quad = \sum_{s=1}^{t}\sum_{i=k}^{-k} q_{|i|}\varepsilon_{s+i} \\
&= q_k\varepsilon_{t+k} + (q_k + q_{k-1})\varepsilon_{t+k-1} + (q_k + q_{k-1} + q_{k-2})\varepsilon_{t+k-2} + \cdots + (q_k + \cdots + q_1)\varepsilon_{t+1} \\
&\quad + (q_k + \ldots + q_0 + q_1 + \ldots + q_k)\varepsilon_t - (q_1 + \cdots + q_k)\varepsilon_t \\
&\quad + (q_k + \cdots + q_0 + \cdots + q_k)\varepsilon_{t-1} - (q_2 + \cdots + q_k)\varepsilon_{t-1} \\
&\quad + \cdots + \\
&\quad + (q_k + \cdots + q_0 + \cdots + q_k)\varepsilon_{t-k} \\
&\quad + \cdots + \\
&\quad + (q_k + \cdots + q_0 + \cdots + q_k)\varepsilon_1 - (q_k + \cdots + q_1)\varepsilon_1 \\
&\quad + (q_1 + \cdots + q_k)\varepsilon_0 + (q_2 + \cdots + q_k)\varepsilon_{-1} + \cdots + (q_{k-1} + q_k)\varepsilon_{-k+2} + q_k\varepsilon_{-k+1} \\
&= q(1)\sum_{s=1}^{t}\varepsilon_s + \sum_{j=1}^{k}\varepsilon_{t+j}\sum_{i=j}^{k} q_i - \sum_{j=0}^{k-1}\varepsilon_{t-j}\sum_{i=j+1}^{k} q_i - \sum_{j=1}^{k}\varepsilon_j\sum_{i=j}^{k} q_i \qquad \text{(A.19)}
\end{aligned}
$$

where $q(1) = q_0 + 2q_1 + \ldots + 2q_k = 1$ since the weights of the symmetric X-11 filter sum to unity, and we also use the assumption $\varepsilon_j = 0, j \le 0$.

**Proof of Proposition 4**

Using (A.19), it can be seen that

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T} y_{t-1}^f\varepsilon_t &= T^{-1}\sum_{t=1}^{T}\left(\sum_{s=1}^{t-1}\varepsilon_s + \sum_{j=1}^{k}\varepsilon_{t-1+j}\sum_{i=j}^{k} q_i - \sum_{j=0}^{k-1}\varepsilon_{t-1-j}\sum_{i=j+1}^{k} q_i - \sum_{j=1}^{k}\varepsilon_j\sum_{i=j}^{k} q_i\right)\varepsilon_t \\
&= T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varepsilon_s\varepsilon_t + T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\varepsilon_{t-1+j}\sum_{i=j}^{k} q_i\right)\varepsilon_t - T^{-1}\sum_{t=1}^{T}\left(\sum_{j=0}^{k-1}\varepsilon_{t-1-j}\sum_{i=j+1}^{k} q_i\right)\varepsilon_t \\
&\quad - T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\varepsilon_j\sum_{i=j}^{k} q_i\right)\varepsilon_t. \qquad \text{(A.20)}
\end{aligned}
$$

Standard results (for example, Hamilton, 1994, equation 17.3.26) imply that

$$
T^{-1}\sum_{t=1}^{T}\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s = T^{-1}\sum_{t=1}^{T}\varepsilon_t y_{t-1} \Rightarrow \frac{1}{2}\sigma^2[W(1)^2 - 1]. \qquad \text{(A.21)}
$$

Further, due to white noise $\varepsilon_t$, it is straightforward to see that

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\varepsilon_{t-1+j}\sum_{i=j}^{k} q_i\right)\varepsilon_t &= T^{-1}\sum_{t=1}^{T}\left(\varepsilon_t^2\sum_{i=1}^{k} q_i + \sum_{j=2}^{k}\varepsilon_{t-1+j}\varepsilon_t\sum_{i=j}^{k} q_i\right) \\
&\Rightarrow \sigma^2\sum_{i=1}^{k} q_i
\end{aligned}
$$

while

$$
T^{-1}\sum_{t=1}^{T}\left(\sum_{j=0}^{k-1}\varepsilon_{t-1-j}\sum_{i=j+1}^{k} q_i\right)\varepsilon_t \to 0 \qquad T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\varepsilon_j\sum_{i=j}^{k} q_i\right)\varepsilon_t \to 0.
$$

Hence the asymptotic distribution corresponding to (A.20) is

$$
T^{-1}\sum_{t=1}^{T} y_{t-1}^f\varepsilon_t \Rightarrow \frac{1}{2}\sigma^2\left[W(1)^2 - 1\right] + \sigma^2\sum_{i=1}^{k} q_i \qquad \text{(A.22)}
$$

To obtain the asymptotic distribution of the normalized bias for the filtered random walk, note that

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T} y_{t-1}^{f} u_t &= T^{-1}\sum_{t=1}^{T} y_{t-1}^{f}\left(q_k \varepsilon_{t+k} + \cdots + q_0 \varepsilon_t + \cdots + q_k \varepsilon_{t-k}\right) \\
&= T^{-1}\sum_{t=1}^{T}\Big\{\left(q_k y_{t+k-1}^{f}\varepsilon_{t+k} + \cdots + q_0 y_{t-1}^{f}\varepsilon_t + \cdots + q_k y_{t-k-1}^{f}\varepsilon_{t-k}\right) \\
&\quad - \sum_{i=1}^{k} q_i\left(y_{t+i-1}^{f} - y_{t-1}^{f}\right)\varepsilon_{t+k} + \sum_{i=1}^{k} q_i\left(y_{t-1}^{f} - y_{t-i-1}^{f}\right)\varepsilon_{t-i}\Big\} \qquad \text{(A.23)}
\end{aligned}
$$

Now

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(y_{t+i-1}^{f} - y_{t-1}^{f}\right)\varepsilon_{t+i} &= T^{-1}\sum_{t=1}^{T}\left(\Delta y_{t+i-1}^{f} + \Delta y_{t+i-2}^{f} + \cdots + \Delta y_{t}^{f}\right)\varepsilon_{t+i} \\
&= T^{-1}\sum_{t=1`}^{T}\left\{[q(L)\varepsilon_{t+i-1}]\varepsilon_{t+i} + [q(L)\varepsilon_{t+i-2}]\varepsilon_{t+i} + \cdots + [q(L)\varepsilon_{t}]\varepsilon_{t+i}\right\} \\
&= \sigma^2 \sum_{j=1}^{i} q_j \qquad \text{(A.24)}
\end{aligned}
$$

and, similarly,

$$
T^{-1}\sum_{t=1}^{T}\left(y_{t-1}^{f} - y_{t-i-1}^{f}\right)\varepsilon_{t-i} = T^{-1}\sum_{t=1}^{T}\left(\Delta y_{t-1}^{f} + \Delta y_{t-2}^{f} + \cdots + \Delta y_{t-i}^{f}\right)\varepsilon_{t-i} = \sigma^2 \sum_{j=0}^{i-1} q_j \ . \qquad \text{(A.25)}
$$

Further,

$$
\begin{aligned}
\sum_{i=1}^{k} q_i - \sum_{i=1}^{k} q_i \sum_{j=1}^{i} q_j + \sum_{i=1}^{k} q_i \sum_{j=0}^{i-1} q_j &= \sum_{i=1}^{k} q_i (1 + q_0 - q_i) = \frac{1}{2}(1-q_0)(1+q_0) - \sum_{i=1}^{k} q_i^2 \\
&= \frac{1}{2}\left[1 - q_0^2 - 2\sum_{i=1}^{k} q_i^2\right] = \frac{1}{2}\left[1 - \sum_{i=-k}^{k} q_i^2\right]
\end{aligned}
$$

where we use the symmetry of $q(L)$ and also the relationship

$$
\sum_{i=1}^{k} q_i = \frac{1}{2}[1 - q_0]
$$

which follows from symmetry together with $q(1) = 1$. Therefore, using (A.22), (A.23) satisfies

$$
T^{-1}\sum_{t=1}^{T} y_{t-1}^{f} u_t \Rightarrow \frac{1}{2}\sigma^2\left[W(1)^2 - 1\right] + \frac{1}{2}\sigma^2\left[1 - \sum_{i=-k}^{k} q_i^2\right]. \qquad \text{(A.26)}
$$

The denominator of (22) follows as

$$
T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^2 \Rightarrow \lambda^2 \int W(r)^2\, dr = \sigma^2 \int W(r)^2\, dr \qquad \text{(A.27)}
$$

(Hamilton, 1994, pp.505-506). Using (A.26) and (A.27) then yields (22).

The *t*-ratio for the filtered data is

$$t_{\hat{\alpha}} = T\hat{\alpha} \times \frac{\sqrt{T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^2}}{\sqrt{T^{-1}\sum_{t=1}^{T}\left(\Delta y_t^f - \hat{\alpha} y_{t-1}^f\right)^2}} .$$

Since $T^{-1}\sum_{t=1}^{T}\left(\Delta y_t^f - \hat{\alpha} y_{t-1}^f\right)^2 \Rightarrow \sigma^2 \sum_{i=-k}^{k} q_i^2$, and using (A.26) and (A.27), (23) is easily obtained.

■

With augmentation of the test regression applied to seasonally adjusted data, under the null hypothesis $\alpha = 0$, the "pseudo-true" regression is

$$\Delta y_t^f = \sum_{i=1}^{p} \phi_i^p \Delta y_{t-1}^f + e_t^p \tag{A.28}$$

where, as in (A.3), both the asymptotic AR coefficients $\phi_i^p$ and the corresponding "disturbance" $e_t^p$ depend on the order of autoregressive augmentation, $p$. Suitably amended to relate to the filtered series, (A.4) also continues to apply, so that the coefficients $\phi_i^p$ can be obtained from the autocovariance properties of $u_t$. In an analogous manner to that discussed in the proof of Proposition 2, and due to the noninvertibility of the two-sided moving average process in (19), $e_t^p$ is autocorrelated for all values of $p$. More specifically, it follows from (19) and (A.28) that

$$\begin{aligned} e_t^p &= u_t - \sum_{i=1}^{p} \phi_i^p u_{t-i} \\ &= \left(q_k \varepsilon_{t+k} + \cdots + q_0 \varepsilon_t + \cdots + q_k \varepsilon_{t-k}\right) - \sum_{i=1}^{p} \phi_i^p \left(q_k \varepsilon_{t+k-i} + \cdots + q_0 \varepsilon_{t-i} + \cdots + q_k \varepsilon_{t-k-i}\right) \\ &= \left(\theta_{-k}^p \varepsilon_{t+k} + \cdots + \theta_0^p \varepsilon_t + \cdots + \theta_k^p \varepsilon_{t-k} + \cdots + \theta_{k+p}^p \varepsilon_{t-k-p}\right) = \theta^p(L)\varepsilon_t \end{aligned} \tag{A.29}$$

where $\theta^p(L)$ is a two-sided moving average, with $k + p$ nonzero lags and $k$ nonzero leads; this establishes (24) of the text. For a given data frequency (typically quarterly or monthly) and given $p$, the implied (asymptotic) moving average coefficients of (A.29) can be obtained analytically.

**Proof of Proposition 5**

When the ADF regression for the seasonally adjusted random walk is augmented to order $p$, OLS estimation yields

$$\begin{bmatrix} \hat{\alpha} - 0 \\ \hat{\phi}_1 - \phi_1^p \\ \vdots \\ \hat{\phi}_p - \phi_p^p \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{T}\left(y_{t-1}^f\right)^2 & \sum_{t=1}^{T} y_{t-1}^f \Delta y_{t-1}^f & \cdots & \sum_{t=1}^{T} y_{t-1}^f \Delta y_{t-p}^f \\ \sum_{t=1}^{T} y_{t-1}^f \Delta y_{t-1}^f & \sum_{t=1}^{T}\left(\Delta y_{t-1}^f\right)^2 & \cdots & \sum_{t=1}^{T} \Delta y_{t-1}^f \Delta y_{t-p}^f \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=1}^{T} y_{t-1}^f \Delta y_{t-p}^f & \sum_{t=1}^{T} \Delta y_{t-1}^f \Delta y_{t-p}^f & \cdots & \sum_{t=1}^{T}\left(\Delta y_{t-p}^f\right)^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^{T} e_t^p y_{t-1}^f \\ \sum_{t=1}^{T} e_t^p \Delta y_{t-1}^f \\ \vdots \\ \sum_{t=1}^{T} e_t^p \Delta y_{t-p}^f \end{bmatrix}$$

The different rates of convergence that apply to the coefficients corresponding to the nonstationary and stationary regressors leads us to consider

$$\begin{bmatrix} T\hat{\alpha} \\ T^{\frac{1}{2}}\left(\hat{\phi}-\phi^{p}\right) \end{bmatrix} = \begin{bmatrix} T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^{2} & h' \\ h & \hat{\Gamma} \end{bmatrix}^{-1} \times \begin{bmatrix} T^{-1}\sum_{t=1}^{T} e_{t}^{p} y_{t-1}^{f} \\ \hat{b} \end{bmatrix}$$

where the ($p \times 1$) vector $h$ has $i^{th}$ element $T^{-\frac{3}{2}}\sum_{t=1}^{T} y_{t-1}^{f}\Delta y_{t-i}^{f} \to 0$ and the ($p \times 1$) vector $\hat{b}$ has $i^{th}$ element $\sum_{t=1}^{T} e_t^p \Delta y_{t-i}^f$, while $\hat{\Gamma}$ is the estimated covariance matrix to order $p$ for $\Delta y_t^f$.

The asymptotic distribution of $T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^{2}$ remains unchanged from (A.27), so that to obtain the distribution of $T\hat{\alpha}$ we need to consider only

$$\begin{aligned} T^{-1}\sum_{t=1}^{T} y_{t-1}^{f} e_{t}^{p} &= T^{-1}\sum_{t=1}^{T} y_{t-1}^{f}\left(\theta_{-k}^{p}\varepsilon_{t+k} + \cdots + \theta_{-1}^{p}\varepsilon_{t+1} + \theta_{0}^{p}\varepsilon_{t} + \theta_{1}^{p}\varepsilon_{t-1} + \cdots + \theta_{k+p}^{p}\varepsilon_{t-k-p}\right) \\ &= T^{-1}\sum_{t=1}^{T}\left\{\left(\theta_{-k}^{p} y_{t+k-1}^{f}\varepsilon_{t+k} + \cdots + \theta_{-1}^{p} y_{t}^{f}\varepsilon_{t+1} + \theta_{0}^{p} y_{t-1}^{f}\varepsilon_{t} + \cdots + \theta_{k+p}^{p} y_{t-k-p-1}^{f}\varepsilon_{t-k-p}\right)\right. \\ &\quad \left. - \sum_{i=1}^{k}\theta_{-i}^{p}\left(y_{t+i-1}^{f} - y_{t-1}^{f}\right)\varepsilon_{t+i} + \sum_{i=1}^{k+p}\theta_{i}^{p}\left(y_{t-1}^{f} - y_{t-i-1}^{f}\right)\varepsilon_{t-i}\right\} \end{aligned}$$

Using (A.22), together with (A.24) and (A.25), we have

$$\begin{aligned} T^{-1}\sum_{t=1}^{T} y_{t-1}^{f} e_{t}^{p} \Rightarrow \frac{1}{2}\sigma^{2}\theta^{p}(1)\left[W(1)^{2}-1\right] + \theta^{p}(1)\sigma^{2}\sum_{i=1}^{k} q_{i} - \sigma^{2}\sum_{i=1}^{k}\theta_{-i}^{p}\sum_{j=1}^{i} q_{j} \\ + \sigma^{2}\sum_{i=1}^{k+1}\theta_{i}^{p}\sum_{j=0}^{i-1} q_{j} + \sigma^{2}\sum_{i=k+2}^{k+p}\theta_{i}^{p}\sum_{j=0}^{k} q_{j}. \end{aligned} \qquad \text{(A.30)}$$

This expression, together with (A.27), yields the asymptotic distribution given in (25).

The corresponding $t$-ratio is given by

$$t_{\hat{\alpha}} = T\hat{\alpha} \times \frac{\sqrt{T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^{2}}}{\sqrt{T^{-1}\sum_{t=1}^{T}\left(\Delta y_{t}^{f} - \hat{\alpha} y_{t-1}^{f} - \sum_{j=1}^{k}\hat{\phi}_{i}\Delta y_{t-j}^{f}\right)^{2}}}$$

and (26) is obtained by noting that since $\hat{\alpha} \to 0$ and $\hat{\phi}_i \to \phi_i^p$, then

$$\begin{aligned} t_{\hat{\alpha}} &\Rightarrow \frac{1}{2}\frac{\sigma^{2}\theta^{p}(1)\left[W(1)^{2}-1\right] + 2\sigma^{2}\left(\theta^{p}(1)\sum_{i=1}^{k} q_{i} - \sum_{i=1}^{k}\theta_{-i}^{p}\sum_{j=1}^{i} q_{j} + \sum_{i=1}^{k+1}\theta_{i}^{p}\sum_{j=0}^{i-1} q_{j} + \sum_{i=k+2}^{k+p}\theta_{i}^{p}\sum_{j=0}^{k} q_{j}\right)}{\sqrt{\sigma^{2}\int W(r)^{2}dr}\sqrt{Var(e_{t}^{p})}} \\ &= \frac{1}{2}\frac{\theta^{p}(1)\left[W(1)^{2}-1\right] + 2\left(\theta^{p}(1)\sum_{i=1}^{k} q_{i} - \sum_{i=1}^{k}\theta_{-i}^{p}\sum_{j=1}^{i} q_{j} + \sum_{i=1}^{k+1}\theta_{i}^{p}\sum_{j=0}^{i-1} q_{j} + \sum_{i=k+2}^{k+p}\theta_{i}^{p}\sum_{j=0}^{k} q_{j}\right)}{\sqrt{\int W(r)^{2}dr}\sqrt{\sum_{i=-k}^{k+p}\theta_{i}^{2}}}. \end{aligned}$$

■

**Proof of Proposition 6**

As $u_t = \sum_{i=k}^{-k} q_{|i|}\varepsilon_{t+i}$, then $\gamma_0 = E\left[u_t^2\right] = \sigma^2 \sum_{i=-k}^{k} q_i^2$ and $\gamma_s = E\left[u_t u_{t-s}\right] = \sigma^2 \sum_{i=-k}^{k-s} q_j q_{j-s}$. Therefore,

in this case, (14) becomes

$$s_u^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 \to \gamma_0 = \sigma^2 \sum_{i=-k}^{k} q_i^2$$

$$s_{tl}^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 + 2T^{-1}\sum_{i=1}^{p} w(i,p)\sum_{t=i+1}^{T}\hat{u}_t\hat{u}_{t-i}$$

$$\to \gamma_0 + 2\sum_{i=1}^{p} w(i,p)\gamma_i = \sigma^2 \sum_{i=-k}^{k} q_i^2 + 2\sigma^2 \sum_{i=1}^{p} w(i,p)\sum_{j=-k}^{k-i} q_j q_{j-i}$$

and, substituting these (22), (23) and (A.27) into (12) and (13), (27) and (28) are easily obtained.

**Table 1.**

**The Null Distribution of the Dickey-Fuller *t*-Statistic in the Presence of a Noninvertible Moving Average**

| Quantile | 0.010 | 0.025 | 0.050 | 0.100 | 0.500 | 0.900 | 0.950 | 0.975 | 0.990 | Size |
|---|---|---|---|---|---|---|---|---|---|---|
| DF distribution | -2.565 | -2.248 | -1.958 | -1.626 | -0.510 | 0.860 | 1.260 | 1.635 | 2.030 | 0.051 |
| Noninvertible MA Process with Augmentation | | | | | | | | | | |
| $p = 0$ | -1.770 | -1.518 | -1.300 | -1.055 | -0.059 | 1.646 | 2.151 | 2.607 | 3.119 | 0.005 |
| $p = 1$ | -3.210 | -2.762 | -2.409 | -2.037 | -0.765 | 0.495 | 0.832 | 1.128 | 1.444 | 0.115 |
| $p = 2$ | -2.223 | -1.902 | -1.647 | -1.359 | -0.313 | 1.177 | 1.612 | 1.994 | 2.411 | 0.021 |
| $p = 3$ | -2.905 | -2.500 | -2.186 | -1.828 | -0.640 | 0.661 | 1.026 | 1.335 | 1.665 | 0.081 |
| $p = 4$ | -2.348 | -2.009 | -1.743 | -1.443 | -0.380 | 1.064 | 1.483 | 1.837 | 2.245 | 0.029 |
| $p = 8$ | -2.440 | -2.098 | -1.819 | -1.508 | -0.428 | 0.979 | 1.391 | 1.737 | 2.122 | 0.037 |
| $p = 12$ | -2.486 | -2.133 | -1.849 | -1.535 | -0.447 | 0.944 | 1.358 | 1.691 | 2.077 | 0.039 |
| $p = 16$ | -2.474 | -2.143 | -1.878 | -1.563 | -0.456 | 0.944 | 1.324 | 1.662 | 2.090 | 0.042 |
| $p = 20$ | -2.487 | -2.163 | -1.890 | -1.568 | -0.472 | 0.932 | 1.316 | 1.663 | 2.060 | 0.044 |
| $p = 24$ | -2.470 | -2.177 | -1.888 | -1.575 | -0.472 | 0.927 | 1.299 | 1.637 | 2.048 | 0.043 |

Notes: The quantiles of the empirical distribution of the ADF test *t*-ratio test are based 15,000 replications and a sample size of 4,000 observations. The DF distribution is obtained from a random walk where the innovation is the white noise process $\varepsilon_t \sim N(0, 1)$. The noninvertible MA is an $I(1)$ process where the innovation is given by $u_t = \varepsilon_t + \varepsilon_{t-1}$, $\varepsilon_t \sim N(0, 1)$, and the ADF regression is estimated with no augmentation and augmentation orders $p = 1, 2, 3, 4, 8, 12, 16, 20, 24$.

**Table 2.**

**The Null Distribution of the Phillips-Perron $Z(t_{\hat{\alpha}})$ Statistic in the Presence of a Noninvertible Moving Average**

| Quantile | 0.010 | 0.025 | 0.050 | 0.100 | 0.500 | 0.900 | 0.950 | 0.975 | 0.990 | Size |
|---|---|---|---|---|---|---|---|---|---|---|
| DF distribution | -2.565 | -2.248 | -1.958 | -1.626 | -0.510 | 0.860 | 1.260 | 1.635 | 2.030 | 0.051 |
| Noninvertible MA Process with Autocorrelation Correction | | | | | | | | | | |
| $p = 0$ | -1.782 | -1.538 | -1.336 | -1.072 | -0.063 | 1.666 | 2.161 | 2.609 | 3.127 | 0.005 |
| $p = 1$ | -2.215 | -1.929 | -1.680 | -1.376 | -0.320 | 1.190 | 1.616 | 1.980 | 2.430 | 0.024 |
| $p = 2$ | -2.344 | -2.038 | -1.785 | -1.466 | -0.389 | 1.081 | 1.485 | 1.841 | 2.259 | 0.032 |
| $p = 3$ | -2.403 | -2.094 | -1.834 | -1.507 | -0.417 | 1.032 | 1.429 | 1.780 | 2.196 | 0.038 |
| $p = 4$ | -2.444 | -2.123 | -1.867 | -1.532 | -0.436 | 1.004 | 1.397 | 1.747 | 2.156 | 0.041 |
| $p = 8$ | -2.504 | -2.172 | -1.916 | -1.575 | -0.465 | 0.955 | 1.346 | 1.697 | 2.077 | 0.047 |
| $p = 12$ | -2.541 | -2.196 | -1.935 | -1.590 | -0.475 | 0.940 | 1.330 | 1.671 | 2.056 | 0.048 |
| $p = 16$ | -2.560 | -2.210 | -1.944 | -1.597 | -0.481 | 0.926 | 1.318 | 1.659 | 2.041 | 0.049 |
| $p = 20$ | -2.569 | -2.211 | -1.945 | -1.598 | -0.485 | 0.925 | 1.310 | 1.645 | 2.037 | 0.050 |
| $p = 24$ | -2.569 | -2.219 | -1.946 | -1.601 | -0.486 | 0.925 | 1.312 | 1.643 | 2.045 | 0.050 |

Notes: The quantiles of the empirical distribution of the $Z(t_{\hat{\alpha}})$ test are based 15,000 replications and a sample size of 4,000 observations. The DF distribution is obtained from a random walk where the innovation is the white noise process $\varepsilon_t \sim N(0, 1)$. The noninvertible MA is an $I(1)$ process where the innovation is given by $u_t = \varepsilon_t + \varepsilon_{t-1}$, $\varepsilon_t \sim N(0, 1)$, and the $Z(t_{\hat{\alpha}})$ is statistic is computed for autocorrelation corrections to order $p$ = 1, 2, 3, 4, 8, 12, 16, 20, 24 using the Bartlett spectral window.

**Table 3.**

**Scaling and Shift Terms for the Augmented Dickey-Fuller Statistics applied to a Seasonally Adjusted Random Walk**

| Augmentation | Numerator shift | Numerator scaling | Denominator scaling | Ratio of scalings | Scaled Shift |
|---|---|---|---|---|---|
| $p = 0$ | 0.174 | 1.000 | 0.909 | 1.100 | 0.191 |
| $p = 1$ | 0.043 | 0.929 | 0.907 | 1.024 | 0.048 |
| $p = 2$ | -0.063 | 0.874 | 0.905 | 0.966 | -0.070 |
| $p = 3$ | -0.188 | 0.814 | 0.903 | 0.901 | -0.209 |
| $p = 4$ | 0.223 | 0.998 | 0.879 | 1.135 | 0.254 |
| $p = 8$ | 0.227 | 0.969 | 0.848 | 1.143 | 0.268 |
| $p = 12$ | 0.220 | 0.936 | 0.818 | 1.143 | 0.269 |
| $p = 16$ | 0.178 | 0.892 | 0.798 | 1.118 | 0.223 |
| $p = 20$ | 0.161 | 0.867 | 0.782 | 1.108 | 0.206 |
| $p = 40$ | 0.100 | 0.789 | 0.738 | 1.070 | 0.135 |
| $p = 100$ | -0.002 | 0.704 | 0.695 | 1.012 | -0.002 |
| $p = 200$ | 0.002 | 0.681 | 0.680 | 1.002 | 0.003 |

Notes: The scaling and shift terms relate to seasonal adjustment of a random walk, with the test regression augmented to order *p*; see (25) and (26). The numerator shift term is defined by

$$2\left(\theta^p(1)\sum_{i=1}^{k} q_i - \sum_{i=1}^{k}\theta_{-i}^p\sum_{j=1}^{i} q_j + \sum_{i=1}^{k+1}\theta_i^p\sum_{j=0}^{i-1} q_j + \sum_{i=k+2}^{k+p}\theta_i^p\sum_{j=0}^{k} q_j\right),$$

the numerator scaling is given by $\theta^p(1)$ and the denominator scaling by $\sqrt{\sum_{i=-k}^{k+p}(\theta_i^p)^2}$ . The ratio of scalings presents the ratio of the numerator to the denominator scaling, while the scaled shift is the numerator shift divided by the denominator scaling.

**Table 4.**

**Quantiles and Size of the Augmented Dickey-Fuller *t*-Statistic for a Seasonally Adjusted Random Walk**

| Quantile | 0.010 | 0.025 | 0.050 | 0.100 | 0.500 | 0.900 | 0.950 | 0.975 | 0.990 | Size |
|---|---|---|---|---|---|---|---|---|---|---|
| DF distribution | -2.628 | -2.220 | -1.944 | -1.620 | -0.511 | 0.883 | 1.274 | 1.621 | 1.995 | 0.049 |
| Seasonally Adjusted Random Walk with Augmentation | | | | | | | | | | |
| $p = 0$ | -2.354 | -2.039 | -1.762 | -1.451 | -0.384 | 1.063 | 1.489 | 1.856 | 2.265 | 0.031 |
| $p = 1$ | -2.557 | -2.206 | -1.901 | -1.574 | -0.473 | 0.928 | 1.315 | 1.662 | 2.047 | 0.045 |
| $p = 2$ | -2.698 | -2.346 | -2.023 | -1.683 | -0.544 | 0.816 | 1.182 | 1.518 | 1.900 | 0.058 |
| $p = 3$ | -2.902 | -2.518 | -2.186 | -1.824 | -0.634 | 0.679 | 1.033 | 1.347 | 1.712 | 0.079 |
| $p = 4$ | -2.282 | -1.977 | -1.699 | -1.403 | -0.341 | 1.126 | 1.561 | 1.942 | 2.374 | 0.027 |
| $p = 8$ | -2.261 | -1.956 | -1.693 | -1.390 | -0.331 | 1.150 | 1.580 | 1.946 | 2.370 | 0.025 |
| $p = 12$ | -2.261 | -1.953 | -1.688 | -1.385 | -0.333 | 1.156 | 1.586 | 1.954 | 2.396 | 0.025 |
| $p = 16$ | -2.296 | -1.962 | -1.723 | -1.415 | -0.358 | 1.104 | 1.501 | 1.879 | 2.277 | 0.026 |
| $p = 20$ | -2.313 | -1.983 | -1.731 | -1.430 | -0.366 | 1.094 | 1.481 | 1.857 | 2.276 | 0.028 |
| $p = 40$ | -2.391 | -2.076 | -1.807 | -1.493 | -0.417 | 1.034 | 1.454 | 1.808 | 2.205 | 0.036 |
| $p = 100$ | -2.542 | -2.196 | -1.894 | -1.584 | -0.478 | 0.919 | 1.317 | 1.685 | 2.047 | 0.044 |
| $p = 200$ | -2.507 | -2.179 | -1.882 | -1.581 | -0.473 | 0.935 | 1.343 | 1.681 | 2.083 | 0.043 |

Notes: The quantiles of the empirical distribution of the ADF test *t*-ratio test are based 15,000 replications and a sample size of 4,000 observations, with the test regression augmented to order *p*. The DF distribution is obtained from a random walk where the innovation is the white noise process $\varepsilon_t \sim N(0, 1)$. Seasonal adjustment is applied to a random walk using the linear approximation to the two-sided quarterly X-11 filter, with 50 additional observation generated and discarded from the beginning and end of the sample. The nominal size is 0.05 for all cases.

**Table 5.**

**Scaling and Shift Terms for the Phillips-Perron $Z(t_{\hat{\alpha}})$ Statistic applied to a Seasonally Adjusted Random Walk**

| Augmentation | Shift factor | Scaling factor |
|---|---|---|
| $p = 1$ | 0.115 | 0.885 |
| $p = 2$ | 0.095 | 0.967 |
| $p = 3$ | 0.033 | 0.905 |
| $p = 4$ | -0.035 | 1.035 |
| $p = 8$ | -0.006 | 1.006 |
| $p = 12$ | 0.004 | 0.996 |
| $p = 16$ | 0.010 | 0.990 |
| $p = 20$ | 0.008 | 0.992 |
| $p = 40$ | 0.004 | 0.996 |
| $p = 100$ | 0.001 | 0.999 |
| $p = 200$ | 0.001 | 0.999 |

Notes : The scaling and shift terms relate to seasonal adjustment of a random walk; see (27) and (28), using an autocorrelation correction to order *p* and the Bartlett weights. The shift term is given by

$$\left[1-\sum_{i=-k}^{k} q_i^2\right]-\left(2\sigma^2\sum_{i=1}^{p}[1-i/(p+1)]\sum_{j=-k}^{k-i} q_j q_{j-i}\right)$$

while the scaling factor is

$$\sqrt{\sum_{i=-k}^{k} q_i^2 + 2\sum_{i=1}^{p}[1-i/(p+1)]\sum_{j=-k}^{k-i} q_j q_{j-i}}\ .$$

**Table 6.**

**Quantiles and Size of the Phillips-Perron $Z(t_{\hat{\alpha}})$ Statistic for a Seasonally Adjusted Random Walk**

| Quantile | 0.010 | 0.025 | 0.050 | 0.100 | 0.500 | 0.900 | 0.950 | 0.975 | 0.990 | Size |
|---|---|---|---|---|---|---|---|---|---|---|
| DF distribution | -2.628 | -2.220 | -1.944 | -1.620 | -0.511 | 0.883 | 1.274 | 1.621 | 1.995 | 0.049 |
| Seasonally Adjusted Random Walk with Autocorrelation Correction | | | | | | | | | | |
| $p = 0$ | -2.273 | -1.973 | -1.732 | -1.444 | -0.376 | 1.083 | 1.484 | 1.883 | 2.301 | 0.027 |
| $p = 1$ | -2.374 | -2.049 | -1.802 | -1.504 | -0.418 | 1.009 | 1.406 | 1.783 | 2.211 | 0.033 |
| $p = 2$ | -2.447 | -2.116 | -1.861 | -1.555 | -0.458 | 0.952 | 1.338 | 1.710 | 2.120 | 0.039 |
| $p = 3$ | -2.538 | -2.189 | -1.922 | -1.612 | -0.497 | 0.891 | 1.268 | 1.628 | 2.034 | 0.047 |
| $p = 4$ | -2.485 | -2.148 | -1.889 | -1.579 | -0.475 | 0.926 | 1.310 | 1.671 | 2.078 | 0.042 |
| $p = 8$ | -2.498 | -2.170 | -1.905 | -1.595 | -0.485 | 0.911 | 1.293 | 1.654 | 2.057 | 0.045 |
| $p = 12$ | -2.505 | -2.175 | -1.911 | -1.601 | -0.488 | 0.900 | 1.290 | 1.655 | 2.053 | 0.046 |
| $p = 16$ | -2.515 | -2.177 | -1.915 | -1.603 | -0.488 | 0.902 | 1.284 | 1.657 | 2.049 | 0.046 |
| $p = 20$ | -2.506 | -2.176 | -1.916 | -1.603 | -0.490 | 0.900 | 1.288 | 1.669 | 2.049 | 0.046 |
| $p = 40$ | -2.534 | -2.183 | -1.915 | -1.607 | -0.490 | 0.907 | 1.291 | 1.658 | 2.048 | 0.046 |
| $p = 100$ | -2.507 | -2.184 | -1.909 | -1.604 | -0.489 | 0.916 | 1.326 | 1.702 | 2.143 | 0.046 |
| $p = 200$ | -2.464 | -2.165 | -1.908 | -1.605 | -0.492 | 0.953 | 1.398 | 1.752 | 2.259 | 0.045 |

Notes: The quantiles of the empirical distribution of the $Z(t_{\hat{\alpha}})$ test are based 15,000 replications and a sample size of 4,000 observations, with an autocorrelation correction applied to order $p$ using Bartlett weights. The DF distribution is obtained from a random walk where the innovation is the white noise process $\varepsilon_t \sim N(0, 1)$. Seasonal adjustment is applied to a random walk using the linear approximation to the two-sided quarterly X-11 filter, with 50 additional observation generated and discarded from the beginning and end of the sample. The nominal size is 0.05 for all cases.

**Table 7.**

**Size of ADF *t*-Statistic for 200 Observations using Unfiltered (*u*) and Filtered (*f*) Data**

| $\theta$ | $\theta_s$ | $\phi_s$ | | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 8$ | $p = 12$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | *u* | 0.050 | 0.049 | 0.048 | 0.049 | 0.046 | 0.048 | 0.043 |
| | | | *f* | 0.035 | 0.045 | 0.056 | 0.081 | 0.029 | 0.030 | 0.028 |
| 0.5 | 0 | 0 | *u* | 0.023 | 0.085 | 0.039 | 0.059 | 0.046 | 0.048 | 0.045 |
| | | | *f* | 0.022 | 0.092 | 0.050 | 0.059 | 0.030 | 0.028 | 0.027 |
| -0.5 | 0 | 0 | *u* | 0.592 | 0.224 | 0.109 | 0.074 | 0.062 | 0.049 | 0.044 |
| | | | *f* | 0.481 | 0.161 | 0.083 | 0.132 | 0.049 | 0.038 | 0.036 |
| 0 | 0 | 0.5 | *u* | 0.451 | 0.162 | 0.076 | 0.023 | 0.060 | 0.046 | 0.042 |
| | | | *f* | 0.207 | 0.068 | 0.054 | 0.064 | 0.039 | 0.032 | 0.031 |
| | | 0.8 | *u* | 0.694 | 0.304 | 0.139 | 0.025 | 0.069 | 0.049 | 0.043 |
| | | | *f* | 0.167 | 0.067 | 0.054 | 0.056 | 0.043 | 0.036 | 0.034 |
| | | 0.9 | *u* | 0.861 | 0.504 | 0.258 | 0.026 | 0.066 | 0.051 | 0.049 |
| | | | *f* | 0.157 | 0.067 | 0.051 | 0.052 | 0.046 | 0.037 | 0.039 |
| | 0.5 | 0.5 | *u* | 0.643 | 0.258 | 0.123 | 0.024 | 0.059 | 0.040 | 0.046 |
| | | | *f* | 0.245 | 0.088 | 0.054 | 0.035 | 0.051 | 0.030 | 0.034 |
| | | 0.8 | *u* | 0.889 | 0.535 | 0.276 | 0.029 | 0.062 | 0.047 | 0.046 |
| | | | *f* | 0.218 | 0.086 | 0.055 | 0.038 | 0.053 | 0.039 | 0.038 |
| | | 0.9 | *u* | 0.970 | 0.748 | 0.493 | 0.028 | 0.064 | 0.045 | 0.048 |
| | | | *f* | 0.205 | 0.088 | 0.053 | 0.032 | 0.050 | 0.035 | 0.038 |
| | -0.5 | 0.8 | *u* | 0.442 | 0.147 | 0.072 | 0.028 | 0.056 | 0.046 | 0.052 |
| | | | *f* | 0.210 | 0.068 | 0.050 | 0.084 | 0.037 | 0.031 | 0.035 |
| | | 0.9 | *u* | 0.563 | 0.214 | 0.104 | 0.028 | 0.065 | 0.055 | 0.051 |
| | | | *f* | 0.186 | 0.068 | 0.055 | 0.086 | 0.041 | 0.036 | 0.034 |
| 0.5 | 0 | 0.5 | *u* | 0.169 | 0.071 | 0.057 | 0.029 | 0.077 | 0.049 | 0.049 |
| | | | *f* | 0.057 | 0.052 | 0.057 | 0.061 | 0.041 | 0.033 | 0.034 |
| | | 0.8 | *u* | 0.376 | 0.132 | 0.068 | 0.026 | 0.113 | 0.055 | 0.045 |
| | | | *f* | 0.042 | 0.044 | 0.050 | 0.051 | 0.044 | 0.036 | 0.033 |
| | | 0.9 | *u* | 0.598 | 0.248 | 0.109 | 0.027 | 0.131 | 0.054 | 0.042 |
| | | | *f* | 0.041 | 0.051 | 0.054 | 0.054 | 0.048 | 0.034 | 0.030 |
| | 0.5 | 0.5 | *u* | 0.307 | 0.100 | 0.058 | 0.023 | 0.089 | 0.036 | 0.048 |
| | | | *f* | 0.059 | 0.047 | 0.045 | 0.036 | 0.051 | 0.026 | 0.033 |
| | | 0.8 | *u* | 0.633 | 0.266 | 0.119 | 0.028 | 0.119 | 0.037 | 0.049 |
| | | | *f* | 0.056 | 0.051 | 0.049 | 0.038 | 0.064 | 0.030 | 0.036 |
| | | 0.9 | *u* | 0.837 | 0.472 | 0.234 | 0.031 | 0.120 | 0.034 | 0.048 |
| | | | *f* | 0.050 | 0.046 | 0.042 | 0.033 | 0.058 | 0.028 | 0.036 |
| | -0.5 | 0.8 | *u* | 0.139 | 0.062 | 0.047 | 0.026 | 0.069 | 0.055 | 0.044 |
| | | | *f* | 0.047 | 0.041 | 0.049 | 0.075 | 0.029 | 0.029 | 0.027 |
| | | 0.9 | *u* | 0.249 | 0.090 | 0.059 | 0.023 | 0.090 | 0.067 | 0.055 |
| | | | *f* | 0.048 | 0.045 | 0.054 | 0.077 | 0.035 | 0.032 | 0.031 |

**Table 7 (continued)**

| $\theta$ | $\theta_s$ | $\phi_s$ | | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 8$ | $p = 12$ |
|---|---|---|---|---|---|---|---|---|---|---|
| -0.5 | 0 | 0.5 | $u$ | 0.947 | 0.648 | 0.376 | 0.067 | 0.082 | 0.056 | 0.051 |
| | | | $f$ | 0.816 | 0.408 | 0.192 | 0.162 | 0.072 | 0.050 | 0.042 |
| | | 0.8 | $u$ | 0.986 | 0.814 | 0.568 | 0.030 | 0.062 | 0.049 | 0.047 |
| | | | $f$ | 0.767 | 0.370 | 0.156 | 0.116 | 0.058 | 0.040 | 0.037 |
| | | 0.9 | $u$ | 0.997 | 0.908 | 0.718 | 0.024 | 0.050 | 0.044 | 0.044 |
| | | | $f$ | 0.730 | 0.338 | 0.143 | 0.102 | 0.047 | 0.040 | 0.038 |
| | 0.5 | 0.5 | $u$ | 0.980 | 0.777 | 0.515 | 0.037 | 0.066 | 0.075 | 0.058 |
| | | | $f$ | 0.830 | 0.451 | 0.205 | 0.096 | 0.061 | 0.059 | 0.049 |
| | | 0.8 | $u$ | 1.000 | 0.930 | 0.758 | 0.019 | 0.041 | 0.062 | 0.060 |
| | | | $f$ | 0.789 | 0.398 | 0.170 | 0.056 | 0.043 | 0.050 | 0.047 |
| | | 0.9 | $u$ | 1.000 | 0.976 | 0.885 | 0.020 | 0.036 | 0.052 | 0.049 |
| | | | $f$ | 0.750 | 0.350 | 0.139 | 0.048 | 0.039 | 0.042 | 0.041 |
| | -0.5 | 0.8 | $u$ | 0.938 | 0.624 | 0.355 | 0.087 | 0.088 | 0.055 | 0.046 |
| | | | $f$ | 0.807 | 0.397 | 0.180 | 0.207 | 0.075 | 0.046 | 0.038 |
| | | 0.9 | $u$ | 0.967 | 0.707 | 0.437 | 0.063 | 0.092 | 0.053 | 0.046 |
| | | | $f$ | 0.784 | 0.370 | 0.172 | 0.197 | 0.078 | 0.044 | 0.041 |

Notes: The DGP is $y_t^{ns} = y_{t-1}^{ns} + \varepsilon_t^{ns} + \theta\varepsilon_{t-1}^{ns}$, $y_t^{s} = \phi_s y_{t-4}^{ns} + \varepsilon_t^{ns} + \theta_s \varepsilon_{t-4}^{ns}$. The ADF test regression includes an intercept and is augmented to order $p$. Results are based on 5,000 replications and a sample size of 200 observations. Filtering applies the linear approximation to the two-sided quarterly X-11 seasonal adjustment filter, with 50 additional observation generated and discarded at the beginning and end of the sample. The nominal size in all cases is 0.050.

**Table 8.**
**Size of PP $Z(t_{\hat{\alpha}})$ Statistic for 200 Observations using Unfiltered (*u*) and Filtered (*f*) Data**

| $\theta$ | $\theta_s$ | $\phi_s$ | | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 8$ | $p = 12$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | *u* | 0.048 | 0.048 | 0.050 | 0.050 | 0.054 | 0.056 |
| | | | *f* | 0.038 | 0.041 | 0.046 | 0.042 | 0.044 | 0.044 |
| 0.5 | 0 | 0 | *u* | 0.028 | 0.032 | 0.035 | 0.035 | 0.035 | 0.033 |
| | | | *f* | 0.025 | 0.029 | 0.034 | 0.034 | 0.034 | 0.032 |
| -0.5 | 0 | 0 | *u* | 0.445 | 0.401 | 0.398 | 0.404 | 0.456 | 0.506 |
| | | | *f* | 0.336 | 0.301 | 0.331 | 0.301 | 0.349 | 0.397 |
| 0 | 0 | 0.5 | *u* | 0.316 | 0.287 | 0.248 | 0.288 | 0.332 | 0.374 |
| | | | *f* | 0.143 | 0.130 | 0.138 | 0.131 | 0.152 | 0.172 |
| | | 0.8 | *u* | 0.551 | 0.509 | 0.429 | 0.512 | 0.562 | 0.607 |
| | | | *f* | 0.114 | 0.105 | 0.107 | 0.105 | 0.119 | 0.136 |
| | | 0.9 | *u* | 0.743 | 0.703 | 0.621 | 0.705 | 0.752 | 0.788 |
| | | | *f* | 0.111 | 0.101 | 0.102 | 0.102 | 0.113 | 0.127 |
| | 0.5 | 0.5 | *u* | 0.490 | 0.448 | 0.378 | 0.449 | 0.504 | 0.550 |
| | | | *f* | 0.161 | 0.145 | 0.134 | 0.143 | 0.168 | 0.195 |
| | | 0.8 | *u* | 0.770 | 0.735 | 0.650 | 0.735 | 0.781 | 0.814 |
| | | | *f* | 0.152 | 0.133 | 0.121 | 0.131 | 0.153 | 0.179 |
| | | 0.9 | *u* | 0.917 | 0.897 | 0.846 | 0.900 | 0.921 | 0.940 |
| | | | *f* | 0.137 | 0.120 | 0.108 | 0.118 | 0.145 | 0.165 |
| | -0.5 | 0.8 | *u* | 0.309 | 0.278 | 0.242 | 0.280 | 0.323 | 0.360 |
| | | | *f* | 0.139 | 0.127 | 0.142 | 0.132 | 0.150 | 0.171 |
| | | 0.9 | *u* | 0.404 | 0.369 | 0.309 | 0.371 | 0.418 | 0.465 |
| | | | *f* | 0.121 | 0.113 | 0.124 | 0.114 | 0.131 | 0.148 |
| 0.5 | 0 | 0.5 | *u* | 0.116 | 0.107 | 0.088 | 0.108 | 0.122 | 0.136 |
| | | | *f* | 0.047 | 0.051 | 0.054 | 0.053 | 0.056 | 0.058 |
| | | 0.8 | *u* | 0.260 | 0.236 | 0.170 | 0.235 | 0.272 | 0.309 |
| | | | *f* | 0.040 | 0.043 | 0.045 | 0.045 | 0.047 | 0.047 |
| | | 0.9 | *u* | 0.456 | 0.414 | 0.324 | 0.414 | 0.465 | 0.516 |
| | | | *f* | 0.044 | 0.045 | 0.047 | 0.047 | 0.050 | 0.050 |
| | 0.5 | 0.5 | *u* | 0.206 | 0.183 | 0.138 | 0.183 | 0.215 | 0.248 |
| | | | *f* | 0.051 | 0.052 | 0.050 | 0.055 | 0.060 | 0.063 |
| | | 0.8 | *u* | 0.482 | 0.438 | 0.341 | 0.437 | 0.492 | 0.540 |
| | | | *f* | 0.052 | 0.053 | 0.050 | 0.055 | 0.060 | 0.060 |
| | | 0.9 | *u* | 0.717 | 0.670 | 0.578 | 0.674 | 0.723 | 0.767 |
| | | | *f* | 0.055 | 0.056 | 0.053 | 0.058 | 0.060 | 0.061 |
| | -0.5 | 0.8 | *u* | 0.115 | 0.105 | 0.094 | 0.107 | 0.121 | 0.135 |
| | | | *f* | 0.053 | 0.056 | 0.063 | 0.058 | 0.064 | 0.067 |
| | | 0.9 | *u* | 0.166 | 0.149 | 0.114 | 0.148 | 0.172 | 0.197 |
| | | | *f* | 0.048 | 0.053 | 0.056 | 0.054 | 0.056 | 0.056 |

**Table 8 (continued)**

| $\theta$ | $\theta_s$ | $\phi_s$ | | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 8$ | $p = 12$ |
|---|---|---|---|---|---|---|---|---|---|
| -0.5 | 0 | 0.5 | $u$ | 0.849 | 0.822 | 0.792 | 0.823 | 0.862 | 0.889 |
| | | | $f$ | 0.656 | 0.607 | 0.622 | 0.609 | 0.666 | 0.716 |
| | | 0.8 | $u$ | 0.952 | 0.936 | 0.905 | 0.936 | 0.955 | 0.969 |
| | | | $f$ | 0.620 | 0.561 | 0.564 | 0.560 | 0.618 | 0.678 |
| | | 0.9 | $u$ | 0.987 | 0.981 | 0.963 | 0.981 | 0.989 | 0.992 |
| | | | $f$ | 0.577 | 0.527 | 0.530 | 0.524 | 0.580 | 0.641 |
| | 0.5 | 0.5 | $u$ | 0.934 | 0.915 | 0.886 | 0.916 | 0.939 | 0.959 |
| | | | $f$ | 0.697 | 0.642 | 0.631 | 0.643 | 0.701 | 0.747 |
| | | 0.8 | $u$ | 0.992 | 0.987 | 0.974 | 0.987 | 0.992 | 0.995 |
| | | | $f$ | 0.650 | 0.589 | 0.568 | 0.585 | 0.649 | 0.702 |
| | | 0.9 | $u$ | 0.998 | 0.997 | 0.992 | 0.997 | 0.998 | 0.998 |
| | | | $f$ | 0.596 | 0.531 | 0.513 | 0.529 | 0.596 | 0.647 |
| | -0.5 | 0.8 | $u$ | 0.840 | 0.808 | 0.783 | 0.808 | 0.847 | 0.880 |
| | | | $f$ | 0.654 | 0.609 | 0.632 | 0.610 | 0.664 | 0.709 |
| | | 0.9 | $u$ | 0.902 | 0.876 | 0.850 | 0.877 | 0.906 | 0.931 |
| | | | $f$ | 0.645 | 0.596 | 0.623 | 0.597 | 0.655 | 0.699 |

Notes: The DGP is $y_t^{ns} = y_{t-1}^{ns} + \varepsilon_t^{ns} + \theta\varepsilon_{t-1}^{ns}$, $y_t^s = \phi_s y_{t-4}^{ns} + \varepsilon_t^{ns} + \theta_s \varepsilon_{t-4}^{ns}$. The $Z(t_{\hat{\alpha}})$ test regression includes an intercept and an autocorrelation correction is applied to order *p*. Results are based on 5,000 replications and a sample size of 200 observations. Filtering applies the linear approximation to the two-sided quarterly X-11 seasonal adjustment filter, with 50 additional observation generated and discarded at the beginning and end of the sample. The nominal size in all cases is 0.050.