WORKING PAPERS AND STUDIES

ISSN 1725-4825

2006 EDITION

Recent Developments in Benchmarking to Annual Totals in X-12-Arima and at Statistics Canada

Conference on seasonality, seasonal adjustment and their implications for short-term analysis and forecasting

Luxembourg, 10-12 May 2006

EUROPEAN COMMISSION

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General and regional statistics

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Luxembourg: Office for Official Publications of the European Communities, 2006

ISBN 92-79-03422-7
ISSN 1725-4825
Catalogue number: KS-DT-06-023-EN-N

RECENT DEVELOPMENTS IN BENCHMARKING TO ANNUAL TOTALS IN X-12-ARIMA AND AT STATISTICS CANADA

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X-12-ARIMA V03+ includes a spec, called FORCE, to produce the table of seasonally adjusted series with constrained yearly totals. The method is based on a benchmarking methodology developed at Statistics Canada. In X-12-ARIMA, the control totals are restricted to be derived from the raw series. This paper gives the ideas behind the method and also gives guidelines for the selection of the parameters. We also present recent methodological developments toward the development of a more generalized benchmarking procedure, which generalizes the benchmarking method in X-12-ARIMA to include more options. For example, the input series does not need to be seasonally adjusted for benchmarking; control totals can be external, and so, a bias parameter can be estimated. Finally we conclude this paper with areas for further developments. Through the paper, we will provide examples to illustrate the different cases.

KEYWORDS: Binding and non-binding benchmarking, Constrained optimization, Denton method, Linear regression model for benchmarking, Measurement errors.

1 Introduction

We define benchmarking as an adjustment of the level of a sub-annual series $s_t, t = 1, \ldots, T$ using auxiliary annual benchmarks $a_m, m = 1, \ldots, M$. We consider two important issues. The first one is to preserve movement in the sub-annual series as much as possible. The second one is to account for the timeliness of annual benchmarks, in the sense that the benchmarks for the observations at the end of the series may not be available yet. The method we considered is driven by a few parameters:

1. The smoothing parameter $0 \leq \rho \leq 1$ with suggested default values $\rho = 0.9$ for monthly sub-annual series and $\rho = 0.9^3 = 0.729$ for quarterly sub-annual series.

2. The adjustment model parameter $\lambda \in \mathcal{R}$ with default value $\lambda = 1$ for a proportional benchmarking model. Two other choices are $\lambda = 0$ for an additive benchmarking model; and $\lambda = 0.5$ with $\rho = 0$, which is pro-rating.

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[2]Susie.Fortier@statcan.ca; contact author for Statistics Canada's ©SAS PROC BENCHMARKING.

3. The bias estimation option, which indicates if a bias parameter is to be estimated or not.

In general, the sub-annual series, also referred to as the indicator series, $s_t, t = 1, \ldots, T$, is associated with a series of dates. For this paper, it is enough to assume that the date values take the form *yyyy pp*. The value *yyyy* represents the year. The value *pp* represents the sub-annual period ranging from 1 to 12 for monthly sub-annual series, or 1 to 4 for quarterly sub-annual series. It is assumed that the dates are mapped into the set of integers 1 to T.

Dates are also necessary for the annual benchmarks $a_m, m = 1, \ldots, M$. Annual benchmarks have a starting date $t_{1,m}$ and an ending date $t_{2,m}$ such that $1 \leq t_{1,m} \leq t_{2,m} \leq T$. With this notation, an *annual benchmark* covers $t_{2,m} - t_{1,m} + 1$ consecutive values from $t = t_{1,m}$ to $t_{2,m}$, and so, benchmarks can be aggregates or individual values at arbitrary points along the series.

The benchmarked series, denoted by $\hat{\theta}_t, t = 1, \ldots, T$, is such that $\sum_{t=t_{1,m}}^{t_{2,m}} \hat{\theta}_t = a_m$; $m = 1, \ldots, M$.

A simple matrix notation is very convenient to represent the relation between the sub-annual series and its benchmarks. For this we define the *coverage fractions* and the *temporal sum operator*.

For $m = 1, \ldots, M$, define the *coverage fractions* $j_{m,t}, t = 1, \ldots, T$ as:

$$j_{m,t} = \begin{cases} 1 & t_{1,m} \leq t \leq t_{2,m} \\ 0 & \text{otherwise} \end{cases}$$

Define the *temporal sum operator* as the matrix J of dimensions $M \times T$ containing the coverage fractions:

$$J = \begin{bmatrix} j_{1,1} & j_{1,2} & \cdots & j_{1,T} \\ \vdots & \vdots & \ddots & \vdots \\ j_{M,1} & j_{M,2} & \cdots & j_{M,T} \end{bmatrix}.$$

2 Bias Estimation

We define the bias as the expected discrepancy between an annual benchmark and its related sub-annual values. Let $1_M = (1, \ldots, 1)'$ be a $M \times 1$ vector of 1; $1_T = (1, \ldots, 1)'$ be a $T \times 1$ vector of 1; $a = (a_1, \ldots, a_M)'$; $s = (s_1, \ldots, s_T)'$. A consistent estimate of the bias is the average discrepancy:

$$b = \frac{\sum_{m=1}^{M} a_m - \sum_{m=1}^{M} \sum_{t=t_{1,m}}^{t_{2,m}} s_t}{\sum_{m=1}^{M} \sum_{t=t_{1,m}}^{t_{2,m}} 1} = \frac{1_M'(a - Js)}{1_M' J 1_T}. \tag{1}$$

When it is more convenient to express the bias in term of a ratio instead of a difference in levels, the bias parameter can be estimated as

$$b = \frac{\sum_{m=1}^{M} a_m}{\sum_{m=1}^{M} \sum_{t=t_{1,m}}^{t_{2,m}} s_t} = \frac{1_M' a}{1_M' J s}. \tag{2}$$

Once a bias parameter is estimated, one can decide to apply it or not. One can even think of estimating the bias parameter with only a few of the most recent benchmarks. Let c be the final value of the bias correction factor. Let $s_t^\dagger = c + s_t$ when the bias correction factor is expressed in the term of a difference in the levels such as in Equation (1), or let $s_t^\dagger = c \cdot s_t$ when c is expressed as a ratio such as in Equation (2). The series $s_t^\dagger$ is called the *re-scaled sub-annual series*.

The rationale for estimating a bias parameter with this simple method is provided in Dagum and Cholette (2006, Chap. 6).

3 The `target` and `start` Arguments of the X-12-ARIMA FORCE Spec

In the case of X-12-ARIMA, the bias option does not exist, and hence $s_t^\dagger = s_t$. Instead, the seasonally adjusted (SA) series s_t is benchmarked to annual control totals derived from the corresponding raw series. The `target` argument specifies which series, say x_t, is used as the target for forcing the totals of the seasonally adjusted series. The choices of target are

- the Original series,
- the Calendar adjusted series,
- the Original series adjusted for permanent prior adjustment factors,
- the Original series adjusted for calendar and permanent prior adjustment factors.

By default, the FORCE spec implies that the calendar year totals in the SA series will be made equal to the calendar year totals of the target series. An alternative starting period for the annual total can be specified with the `start` argument; consequently, annual total starting at any other period other than that specified by the `start` argument may not be equal. This will be illustrated in Figure 7 in Section 5.2.

Notation-wise, the modifications are as follows. The annual benchmarks a_m are derived from the `target` series $x_t, t = 1, \ldots, T$. Let $x = (x_1, \ldots, x_T)'$. If we write $P = 4$ for quarterly series and $P = 12$ for monthly series, then a typical row of J will take the form $(0, \ldots, 0, 1_P', 0, \ldots, 0)$. The vector of benchmarks is $a = Jx$.

4 Benchmarking

Methodological details for the benchmarking formulae in this section are provided in Section 7.

Define C as the $T \times T$ matrix with $|s_t^\dagger|^\lambda$ as the t-th element of the main diagonal and 0 elsewhere[3].

The parameter ρ is used to select the way benchmarked values are computed.

For $\rho < 1$, the benchmarked series $\hat{\theta} = \left(\hat{\theta}_1, \ldots, \hat{\theta}_T\right)$ is:

$$\hat{\theta} = s^\dagger + V_e J' V_d^{-1} \left(a - J s^\dagger\right). \tag{3}$$

In this equation, $V_d = J V_e J'$; $V_e = C \Omega_e C$; and Ω_e is the $T \times T$ matrix[4] defined by $\Omega_e = ((\rho^{|i-j|})), i, j = 1, \ldots, T$.

For $\rho = 1$, the benchmarked series $\hat{\theta}$ is:

$$\hat{\theta} = s^\dagger + W \left(a - J s^\dagger\right). \tag{4}$$

In this equation, W is the $T \times M$ upper-right corner matrix from the following matrix product:

$$\begin{bmatrix} C^{-1}\Delta'\Delta C^{-1} & J' \\ J & 0 \end{bmatrix}^{-1} \begin{bmatrix} C^{-1}\Delta'\Delta C^{-1} & 0 \\ J & I_M \end{bmatrix} = \begin{bmatrix} I_T & W \\ 0 & W_\nu \end{bmatrix};$$

Δ is the $T-1 \times T$ matrix with -1 at index (i, i), 1 at index $(i, i+1)$, $i = 1, \ldots, T-1$, and 0 elsewhere; and I_M is the $M \times M$ Identity matrix. (The $M \times M$ matrix W_ν is associated with the Lagrange multipliers.)

5 Available Benchmarking Programs and Examples

The FORCE spec is available for testing with X-12-ARIMA Version 0.3. Interested persons should write to X12@census.gov for more information.

At Statistics Canada, we are developing an in-house SAS© procedure called PROC BENCHMARKING. Interested persons should write to Susie.Fortier@statcan.ca for more information.

The examples in this section were obtained by running test versions of the programs that were made available to us prior to the conference.

[3]The diagonal elements of the matrix C can be re-scaled to avoid numerical problems. For example, they can be divided by their overall mean.

[4]Define $\rho^0 = 1$. Also, for $\rho = 0$, the matrix Ω_e is the Identity matrix.

5.1 PROC BENCHMARKING

The following statements read a seasonal quarterly time series into a SAS dataset called mySeries, and the corresponding annual benchmarks into a SAS dataset called myBenchmarks.

```
DATA mySeries;
  INPUT @01 year 4.
        @06 period 1.
        @08 value;
  CARDS;
1998 1 1.9
1998 2 2.4
1998 3 3.1
1998 4 2.2
1999 1 2.0
1999 2 2.6
1999 3 3.4
1999 4 2.4
2000 1 2.3
  (... more data);
  RUN;

DATA myBenchmarks;
  INPUT @01 startYear 4.
        @06 startPeriod 1.
        @08 endYear 4.
        @13 endPeriod 1.
        @15 value;
  CARDS;
1998 1 1998 4 10.3
1999 1 1999 4 10.2
  (... more data);
  RUN;
```

The following PROC BENCHMARKING step performs benchmarking by selecting the suggested default values $\lambda = 1$, $\rho = 0.729 = 0.9^3$ and with estimation of the bias parameter according to Equation (2). The benchmarked series is stored in the SAS dataset called outSeries[5].

```
PROC BENCHMARKING
   BENCHMARKS=myBenchmarks       SERIES=mySeries
   OUTBENCHMARKS=outBenchmarks   OUTSERIES=outSeries
   RHO=0.729  LAMBDA=1  BIASOPTION=3;
RUN;
```

[5] The OUTBENCHMARKS option stores the benchmarks that were actually used by the procedure.

The same statements but with BIASOPTION = 1 calculate the benchmarked series without bias estimation. The resulting benchmarked series are displayed in Figure 1. The Benchmarked to Indicator ratios (BI-ratios) are displayed in Figure 2. The step-function represents the BI-ratios corresponding to pro-rating. The main difference in the the BI-ratios with and without bias estimation is at the end with years without benchmarks where the BI-ratios converge to 1 without bias estimation and to the estimated bias value $b = 0.964$ when bias estimation is applied.

Finally, Figure 3 displays the BI-ratios with bias estimation, $\lambda = 1$ and various values of the parameter ρ. When the value of ρ increases to 1, the series of BI-ratios is smoother; conversely, when the value of ρ is close to 0, the BI-ratios converge faster to the estimated bias value (0.964 in this case). This figure also illustrates that the suggested default value of $\rho = 0.9^3$ is a good compromise between smoothing and fast convergence of the BI-ratios at the end of the series.

5.2 X-12-ARIMA FORCE spec

A multiplicative monthly seasonal adjustment is to be performed using automatic outlier identification and ARIMA forecast extension on the Canadian Department Stores Retail Trade Series[6]. Trading-day and Easter adjustment factors are computed from the regression spec. The annual totals of the seasonally adjusted series will be forced to equal the totals in the calendar adjusted series with the regression-based method. The suggested default values[7] are hard-coded in the FORCE spec. The parameters are set to `lambda`= 1 to smooth the ratios (instead of the differences) in the annual totals. The value `rho`=0.9 ensures that the BI-ratios in the incomplete years will converge to 1, which is defined as the theoretical value for the ratio of the calendar year total of the calendar adjusted raw series over that of the seasonally adjusted series.

```
series{... save=a18}
transform{function=log}
regression{ variables=(TD easter[8])}
outlier{ ...}
arima{...}
forecast{...}
x11{... save=d11}
force{
  lambda=1
  rho=0.9
  target=calendaradj
  type=regress
  save=saa }
```

[6] Trade Group 170

[7] The `usefcst` argument was added to the FORCE spec after the conference. With default value `yes`, it determines if forecasts are appended to the series.

Figure 4 displays the seasonally adjusted series (D11) and the corrected seasonally adjusted series (D11A). In this case the indicator series is the seasonally adjusted series and the benchmarks are the annual totals of the calendar adjusted raw series. Not that much can be said from that figure; so, the differences in the two series are displayed in Figure 5, where one can clearly see the effect of the smoothing parameter `rho`=0.9. Figure 6 displays the growth rates in the seasonally adjusted series before and after the FORCE spec. Clearly, benchmarking did not affect the growth rates in any noticeable way, which was expected given the differences displayed in Figure 5.

Finally, as discussed earlier, Figure 7 shows that the FORCE spec implies that the calendar year totals in the seasonally adjusted series are made equal to those of the calendar adjusted raw series. Annual total starting at any other period other than January are not at all equal, and so, contrary to popular belief, benchmarking a seasonally adjusted series to annual totals via the methodology implemented in the FORCE spec does not produce a seasonally adjusted series with constant seasonal factors.

6 Guidelines

The following guidelines are mainly based on Chen and Wu (2003).

When using the FORCE spec of X-12-ARIMA:

1. Use `lambda`= 0 for an additive decomposition model and `lambda`= 1 for a multiplicative decomposition model.
2. Use `rho`= 0.9 for monthly time series or `rho`=0.729 for quarterly series.
3. Avoid using `type=denton` or `type=regress` with `rho`= 1. In general, benchmarked values in incomplete years are less accurate, and the program involves the inversion of a much larger matrix. Consequently, users are to expect a significant increase in computing time when using `rho`= 1.
4. An alternative to using `rho`= 1 is thus to use `rho`= 0.999.

The following two guidelines do not apply to X-12-ARIMA.

1. If there are indications that the true autocorrelation of the error is very strong, say $\rho > 0.95$, then use `rho`= 0.98.
2. If possible, use a reasonable method to estimate the autocorrelation structure of the error instead of using the above default values. There will be a gain in accuracy for years without benchmarks. A method is provided in Chen and Wu (2001).

7 Methodological Details

The benchmarked series in Equations (3) and (4) are obtained as the solution of the following minimization problem. For given parameters λ and ρ, and re-scaled series $s_t^\dagger$, find the value $\hat{\theta}_t$ that minimize the following function of θ_t :

$$f(\theta_1,\ldots,\theta_T) = \left(1-\rho^2\right)\left(\frac{s_1^\dagger-\theta_1}{|s_1^\dagger|^\lambda}\right)^2 + \sum_{t=2}^{T}\left[\left(\frac{s_t^\dagger-\theta_t}{|s_t^\dagger|^\lambda}\right) - \rho\left(\frac{s_{t-1}^\dagger-\theta_{t-1}}{|s_{t-1}^\dagger|^\lambda}\right)\right]^2 \tag{5}$$

under the constraints

$$\sum_{t=t_{1,m}}^{t_{2,m}} \theta_t = a_m,\ m = 1,\ldots,M. \tag{6}$$

Some motivation might be needed to justify the choice of the function to be minimized. First, consider the case where $\lambda = 0$ and $\rho = 1$. Then, the function to be minimized under the constraints (6) becomes

$$\begin{aligned} f(\theta_1,\ldots,\theta_T) &= \sum_{t=2}^{T}\left[\left(s_t^\dagger-\theta_t\right) - \left(s_{t-1}^\dagger-\theta_{t-1}\right)\right]^2 \\ &= \sum_{t=2}^{T}\left[\left(s_t^\dagger-s_{t-1}^\dagger\right) - \left(\theta_t-\theta_{t-1}\right)\right]^2, \end{aligned}$$

which aims at preserving the period to period changes in the original series. That is the criterion of Denton (1971) modified by Cholette (1984), called the modified Denton method[8].

Next, consider the case where the re-scaled series is made of positive numbers, $\lambda = 1$, and $\rho = 1$. Then, the function to be minimized under the constraints (6) becomes

$$\begin{aligned} f(\theta_1,\ldots,\theta_T) &= \sum_{t=2}^{T}\left[\left(\frac{s_t^\dagger-\theta_t}{s_t^\dagger}\right) - \left(\frac{s_{t-1}^\dagger-\theta_{t-1}}{s_{t-1}^\dagger}\right)\right]^2 \\ &= \sum_{t=2}^{T}\left[\frac{\theta_t}{s_t^\dagger} - \frac{\theta_{t-1}}{s_{t-1}^\dagger}\right]^2 \end{aligned}$$

which, contrary to popular belief, does not preserve the period-to-period growth rates, but, as explained in Bloem, Dippelsman, and Mæhel (2001), is a variant of the proportional Denton criterion that seeks to minimize the change in the benchmarking revision ratios $\theta_t/s_t^\dagger$ (BI-ratios).

Next, consider the case where the re-scaled series is made of positive numbers, $\lambda = 1/2$, and $\rho = 0$. Then, the function to be minimized under the constraints (6) becomes

$$f(\theta_1,\ldots,\theta_T) = \sum_{t=1}^{T}\left(\frac{s_t^\dagger-\theta_t}{\sqrt{s_t^\dagger}}\right)^2,$$

[8]Originally, Denton puts 1 instead of $(1-\rho^2)$ as the coefficient for the first term in (5).

with solution

$$\hat{\theta}_t = s_t^\dagger \left(\frac{a_m}{\sum_{t=t_{1,m}}^{t_{2,m}} s_t^\dagger} \right), \; t_{1,m} \le t \le t_{2,m}.$$

This is the well-known formula for pro-rating. When $\lambda = 0$ and $\rho = 0$, the function to be minimized under the constraints (6) becomes

$$f(\theta_1, \ldots, \theta_T) = \sum_{t=1}^{T} \left(s_t^\dagger - \theta_t \right)^2,$$

with solution

$$\hat{\theta}_t = s_t^\dagger + \frac{1}{t_{2,m} - t_{1,m} + 1} \left(a_m - \sum_{t=t_{1,m}}^{t_{2,m}} s_t^\dagger \right), \; t_{1,m} \le t \le t_{2,m}.$$

Obviously, in all those special cases, the solution can be computed directly from Equations (3) and (4) with the appropriate values for the parameters λ and ρ.

Figure 3 illustrates how the ratios of the benchmarks to the corresponding totals in the indicator series (the step function corresponding to pro-rating) are smoothed with different values of the parameter ρ. Applying the ratios from pro-rating (without smoothing) creates the so-called *step-problem* between years. The growth rate from the last quarter of a year to the first quarter of the next year in the benchmarked series takes all the effect due to the new benchmark. The variant of the proportional Denton method is obtained with $\rho = 1$. Apart from smoothing the step function, an undesirable feature of this method is that it repeats the last BI-ratio for the observations without a benchmark at the end of the series. For observations without a benchmark, the best estimate of the BI-ratio is the estimated value of the bias; so, repeating the last value is not appropriate. However, to obtain a smooth transition from this last BI-ratio to the bias, one need to have a smooth transition or convergence. This is obtained by having the parameter $\rho < 1$. Clearly, setting $\rho < 1$ produces BI-ratios that converge to the bias parameter at the end of the series. However, for observations with a benchmark, the BI-ratios are closer to those obtained with the proportional Denton method ($\rho = 1$) and smoother when $\rho \to 1$.

We now give the derivation of Equation (3), which we reproduce from Quenneville, Cholette, Huot, Chiu, and Di Fonzo (2005). The function (5) to be minimized is proportional to $\Omega_e^{-1/2} C^{-1} \left(s^\dagger - \theta \right)$. Minimizing this function subject to $J\theta = a$ entails minimizing the function

$$\left(s^\dagger - \theta \right)' \left(C \Omega_e C \right)^{-1} \left(s^\dagger - \theta \right) + 2\nu' \left(J\theta - a \right)$$

with respect to the element of θ and ν, where 2ν is the vector of Lagrange multipliers. Differentiation with respect to these elements leads to the equations

$$\begin{aligned} \left(C \Omega_e C \right)^{-1} \left(s^\dagger - \theta \right) &= J'\nu \\ J\theta &= a. \end{aligned} \tag{7}$$

From Equation (7), $\theta = s^\dagger - C\Omega_e C J'\nu$, and consequently, $J\theta = Js^\dagger - JC\Omega_e CJ'\nu = a$. It follows that $\nu = -\left(JC\Omega_e CJ' \right)^{-1} \left(a - Js^\dagger \right)$, and so $\hat{\theta}$ from Equation (3) follows with $V_e = C\Omega_e C$ and $V_d = JV_e J'$.

8 Future Developments

An implied regression model for benchmarking: The benchmarked series can be computed assuming the following implied regression benchmarking model made of the two linear equations like in Cholette and Dagum (1994). The model is:

$$s^{\dagger} = \theta + e; \; E(e) = 0, \; \mathrm{Cov}(e) = V_e, \tag{8}$$

$$a = J\theta + \epsilon; \; E(\epsilon) = 0, \; \mathrm{Cov}(\epsilon) = V_{\epsilon}, \tag{9}$$

with e and ϵ uncorrelated; and where θ is a constant[9] vector and is regarded as parameters in this regression model.

For further reference, re-write the model defined by Equations (8) and (9) as a standard linear model of the form $y = X\theta + u$ where

$$y = \begin{pmatrix} s^{\dagger} \\ a \end{pmatrix}, \; X = \begin{pmatrix} I \\ J \end{pmatrix}, \; u = \begin{pmatrix} e \\ \epsilon \end{pmatrix} \tag{10}$$

and define the covariance matrix of u as

$$V = \begin{pmatrix} V_e & 0 \\ 0 & V_{\epsilon} \end{pmatrix}. \tag{11}$$

In X-12-ARIMA FORCE spec, the matrix V_{ϵ} is set to 0, the matrix V_e is specified by the choice of the parameter λ and ρ, and the benchmarked values are computed according to Equations (3) or (4). Note that with these choices for the matrices V_e and V_{ϵ}, the "regression model" defined by Equations (8) and (9) is just a way to perform the numerical computations for the minimization of Equation (5) under the constraints defined by Equation (6).

A few generalizations are needed outside the scope of X-12-ARIMA FORCE spec. They are going to be implemented in a future version of PROC BENCHMARKING.

Measurement errors from the infra-annual series: A first generalization is to consider that $V_e = C\Omega_e C$ where C is a diagonal matrix with the standard deviation of the measurement errors, $s_t^{\dagger} - \theta_t$, on its main diagonal, and that Ω_e is the auto-covariance matrix of the corresponding standardized errors (having variance equal to 1). For example, if the errors e have constant coefficients of variation (relative variance), say $cv = 1\%$, then $C = cv \times \mathrm{diag}(|s_t^{\dagger}|)$ represents the standard errors, and $V_e = C\Omega_e C$ represents the covariance matrix of e. In fact, Ω_e can be used to represent the autocorrelation of any stationary ARMA process with unit variance, not just AR(1). Then, Ω_e has a more complicated structure then that we mentioned under Equation (3).

Measurement errors from the benchmarks: A typical application of benchmarking at Statistics Canada involves cases where the benchmarks themselves are subject to errors. Such benchmarks are called non-binding. In this case, the matrix V_{ϵ} in Equation (9) is different from the zero matrix.

[9] See Durbin and Quenneville (1997) and the references therein for the case where θ is a random vector.

Let $\epsilon = (\epsilon_1, \ldots, \epsilon_M)'$ be the vector of measurement errors associated with the benchmarks, and V_ϵ represents its covariance matrix. Then the covariance matrix of the annual discrepancies $a - Js^\dagger$ is $V_d = JV_eJ' + V_\epsilon$.

Write $V_e = \sigma_e^2 C\Omega_e C$, and correspondingly $V_\epsilon = \sigma_\epsilon^2 \Omega_\epsilon$. Then only the ratio $\sigma_\epsilon^2/\sigma_e^2$ will be needed in the benchmarking formulae. For this it will be sufficient to define $V_e = C\Omega_e C$ and $V_d = JV_eJ' + (\sigma_\epsilon^2/\sigma_e^2)\,\Omega_\epsilon$ in Equation (3). So, it is only when the ratio $\sigma_\epsilon^2/\sigma_e^2$ is very close to zero that the measurement errors in the benchmarks can be ignored in V_d.

Variance estimation of the benchmarked values: Let V_e and V_ϵ be known error covariance matrices for e and ϵ, and assume for now that both are positive definite. That is, let $V_e = E\left(s^\dagger - \theta\right)\left(s^\dagger - \theta\right)'$ and similarly for V_ϵ. Then $\hat\theta$ is provided by[10]

$$\begin{aligned}\hat\theta &= \left(X'V^{-1}X\right)^{-1} X'V^{-1}y \\ &= s^\dagger + V_eJ'\left(JV_eJ' + V_\epsilon\right)^{-1}\left(a - Js^\dagger\right) \qquad (13)\end{aligned}$$

and

$$V_{\hat\theta} = \left(X'V^{-1}X\right)^{-1} = V_e - V_eJ'\left(JV_eJ' + V_\epsilon\right)^{-1} JV_e \qquad (14)$$

is the error covariance matrix of $\hat\theta$, that is $E\left(\hat\theta - \theta\right)\left(\hat\theta - \theta\right)'$.

Note that when $V_\epsilon = 0$ (binding benchmarks), Equation (13) is consistent with Equation (3). Note also that Equation (14) holds only if V_e and V_ϵ are made of known constants. In a case such as using $V_e = C\Omega_e C$ with $C = cv \times \mathrm{diag}(|s_t^\dagger|)$, the matrix V_e is only an estimate of the true covariance matrix, then Equation (14) is only an estimate of the true $V_{\hat\theta}$.

Binding benchmarking with benchmarks subject to measurement errors: Due to a variety of reasons, the benchmarked values can always be calculated by putting $V_\epsilon = 0$, i.e. to use the binding benchmarking formulae (3), even when the benchmarks are non-binding. Denote the benchmarked values using the binding benchmarking formulae by $\hat\theta_0$, then the error covariance matrix is:

$$\mathrm{Cov}\left(\hat\theta_0 - \theta\right) = \left[I - V_eJ'\left(JV_eJ'\right)^{-1} J\right] V_e \left[I - J'\left(JV_eJ'\right)^{-1} JV_e\right] + \left(J'V_\epsilon^{-1}J\right)^{-1}. \qquad (15)$$

We are now going to derive Equations (15). For the regression model $y = X\theta + u$ defined by Equation (10), if instead of V,

$$V_\delta = \begin{pmatrix} V_e & 0 \\ 0 & \delta V_\epsilon \end{pmatrix}$$

is the covariance matrix of u, then the generalized least squares estimate of θ is

$$\begin{aligned}\hat\theta_\delta &= \left(X'V_\delta^{-1}X\right)^{-1} X'V_\delta^{-1}y \qquad (16)\\ &= s^\dagger + V_eJ'\left(JV_eJ' + \delta V_\epsilon\right)^{-1}\left(a - Js^\dagger\right). \qquad (17)\end{aligned}$$

[10]To arrive to the displayed expressions, the following matrix inversion formulae must be used:

$$(A^{-1} + BC^{-1}B')^{-1} = A - AB(B'AB + C)^{-1}B'A. \qquad (12)$$

However, only $\hat{\theta}_1$ is the correct result of the non-binding benchmarking; $\hat{\theta}_0$ is the result of using the binding benchmarking formulae.

For any $\delta > 0$ consider,

$$\begin{aligned}\hat{\theta}_\delta - \theta &= \left(X'V_\delta^{-1}X\right)^{-1} X'V_\delta^{-1}(X\theta + u) - \theta \\ &= \left(X'V_\delta^{-1}X\right)^{-1} X'V_\delta^{-1}u.\end{aligned}$$

Since $\mathrm{Cov}(u) = V_1 = V$,

$$\mathrm{Cov}\left(\hat{\theta}_\delta - \theta\right) = \left(X'V_\delta^{-1}X\right)^{-1} X'V_\delta^{-1}VV_\delta^{-1}X\left(X'V_\delta^{-1}X\right)^{-1}. \tag{18}$$

We wish to calculate $\mathrm{Cov}\left(\hat{\theta}_\delta - \theta\right)$ when $\delta \to 0$. For this, consider

$$X'V_\delta^{-1}X = V_e^{-1} + J'(\delta V_\epsilon)^{-1}J$$

and

$$X'V_\delta^{-1} = (V_e^{-1}, J'(\delta V_\epsilon)^{-1}).$$

Then

$$\left(X'V_\delta^{-1}X\right)^{-1} X'V_\delta^{-1} = \left[\left(V_e^{-1} + J'(\delta V_\epsilon)^{-1}J\right)^{-1} V_e^{-1}, \left(V_e^{-1} + J'(\delta V_\epsilon)^{-1}J\right)^{-1} J'(\delta V_\epsilon)^{-1}\right].$$

The first term on the right hand side of this equation simplifies to

$$I - V_eJ'\left(JV_eJ' + \delta V_\epsilon\right)^{-1}J$$

using the matrix inversion formulae (12), and the second term simplifies to

$$\left[\delta V_e^{-1} + J'V_\epsilon^{-1}J\right]^{-1} J'V_\epsilon^{-1}.$$

When $\delta \to 0$,

$$\left(X'V_\delta^{-1}X\right)^{-1} X'V_\delta^{-1} \to \left[I - V_eJ'\left(JV_eJ'\right)^{-1}J, \left(J'V_\epsilon^{-1}J\right)^{-1} J'V_\epsilon^{-1}\right].$$

Substitution of this result into Equation (18) gives the final result (15).

9 Conclusions

This paper presented recent developments in benchmarking at Statistics Canada that led to the new X-12-ARIMA FORCE spec and Statistics Canada ©SAS procedure PROC BENCHMARKING. In top of presenting the methodological details behind those two computer programs, the paper presented guidelines and ideas for future developments.

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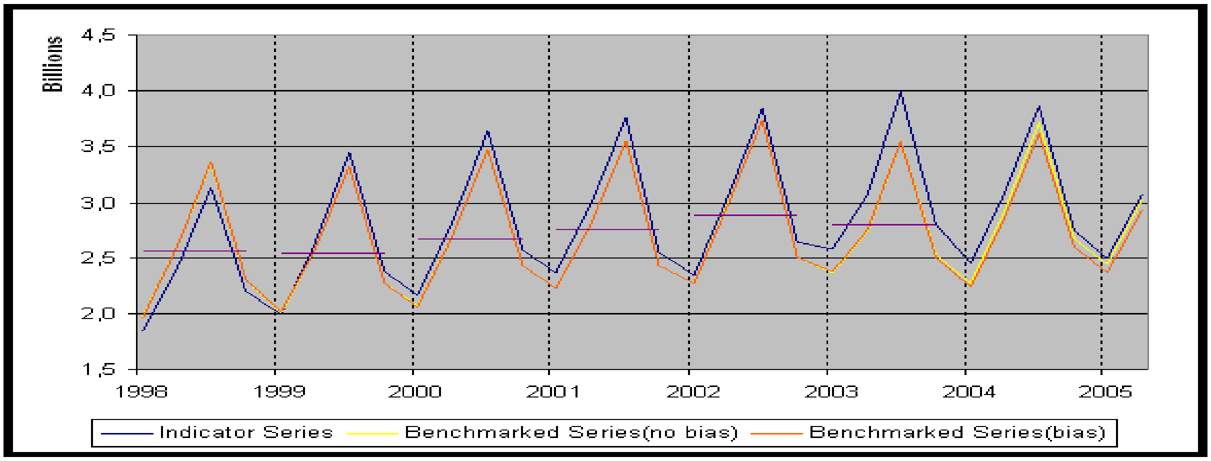

Figure 1: Benchmarked quarterly series with and without bias estimation with $\lambda = 1$ and $\rho = 0.729$. The horizontal lines display the values of the annual benchmarks divided by 4 to re-scale them to the level of the quarterly series.

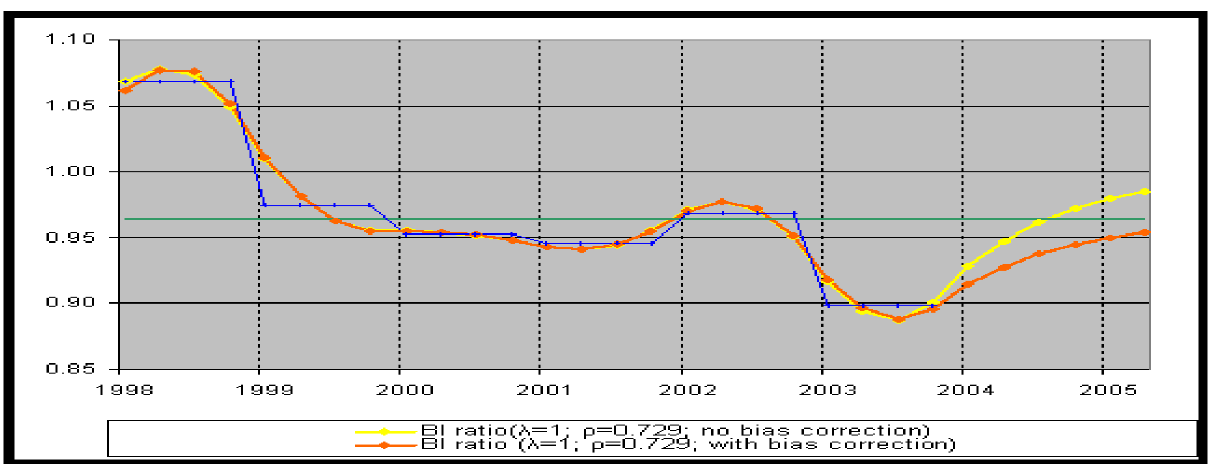

Figure 2: Benchmarked to Indicator ratios with and without bias estimation for $\lambda = 1$ and $\rho = 0.729$. The horizontal line at 0.964 represents the estimated bias value.

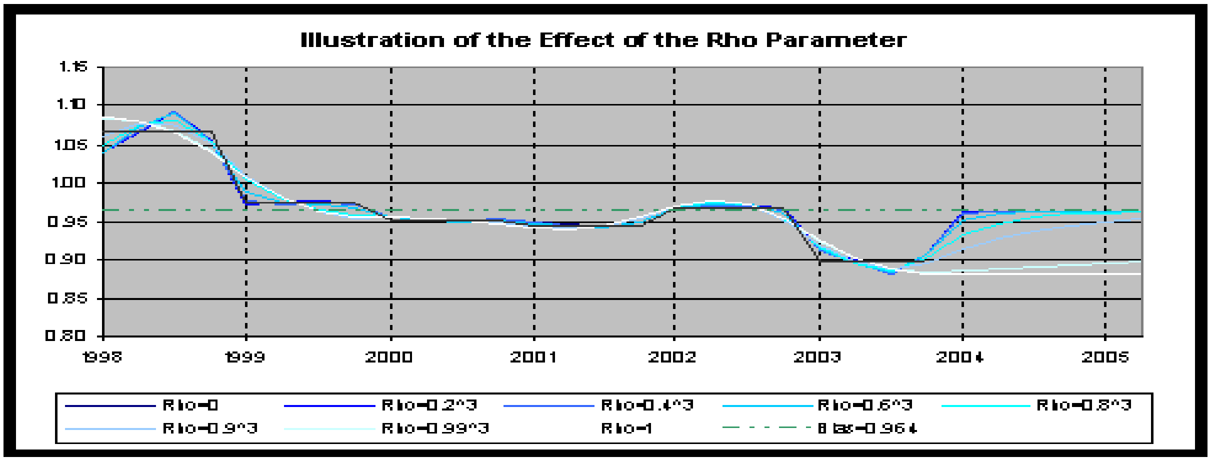

Figure 3: Benchmarked to Indicator ratios with bias estimation for $\lambda = 1$ and various values of ρ.

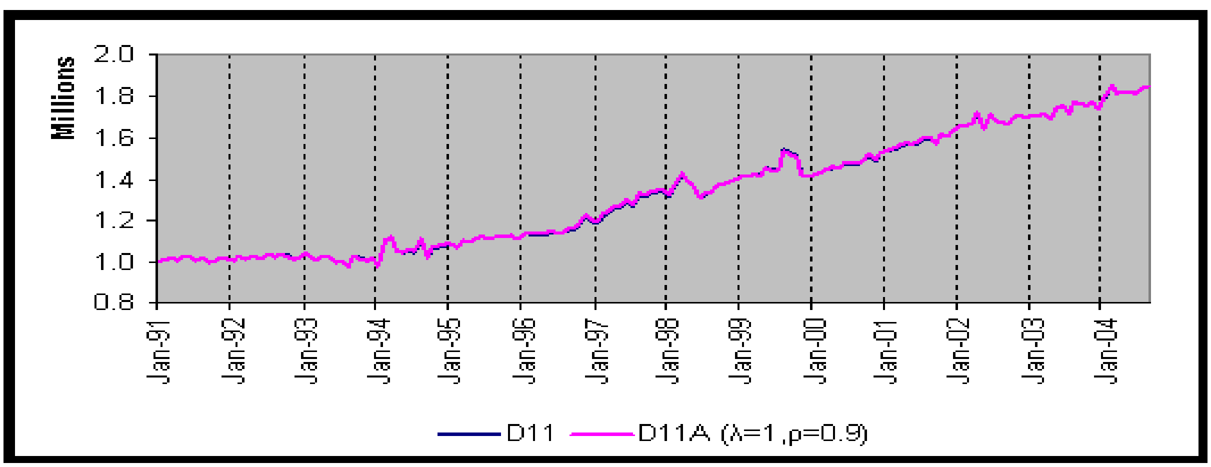

Figure 4: Canadian Department Stores Retail Trade Sales: seasonally adjusted series (D11) and corrected seasonally adjusted series (D11A) to match the annual totals in the calendar adjusted raw series.

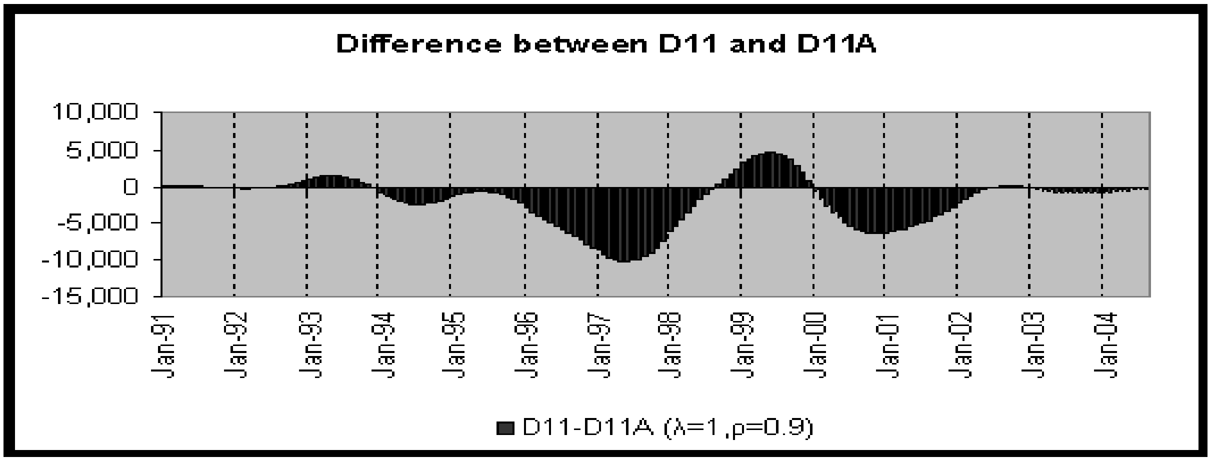

Figure 5: Canadian Department Stores Retail Trade Sales: differences between D11 and D11A.

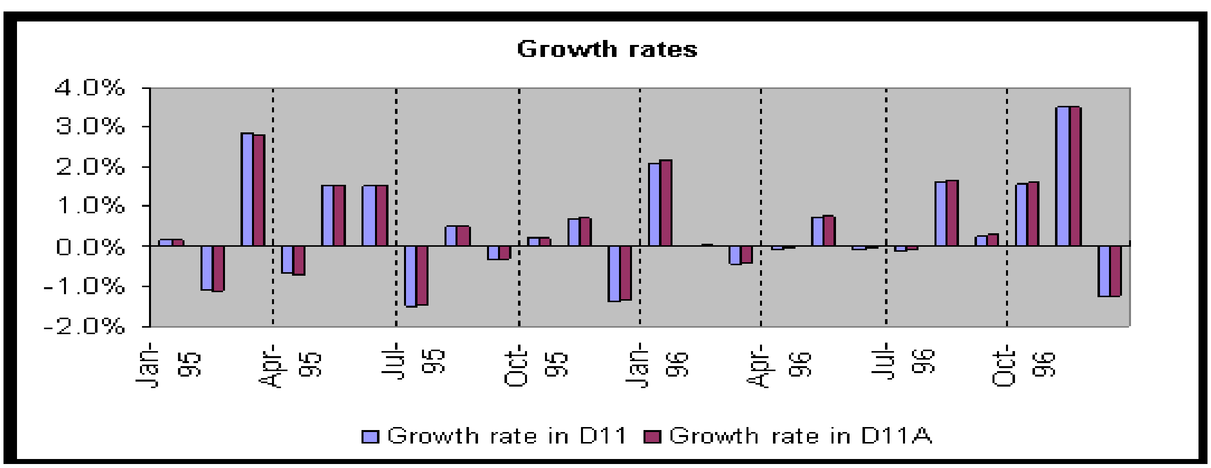

Figure 6: Canadian Department Stores Retail Trade Sales: Growth rates before and after the X-12-ARIMA FORCE spec.

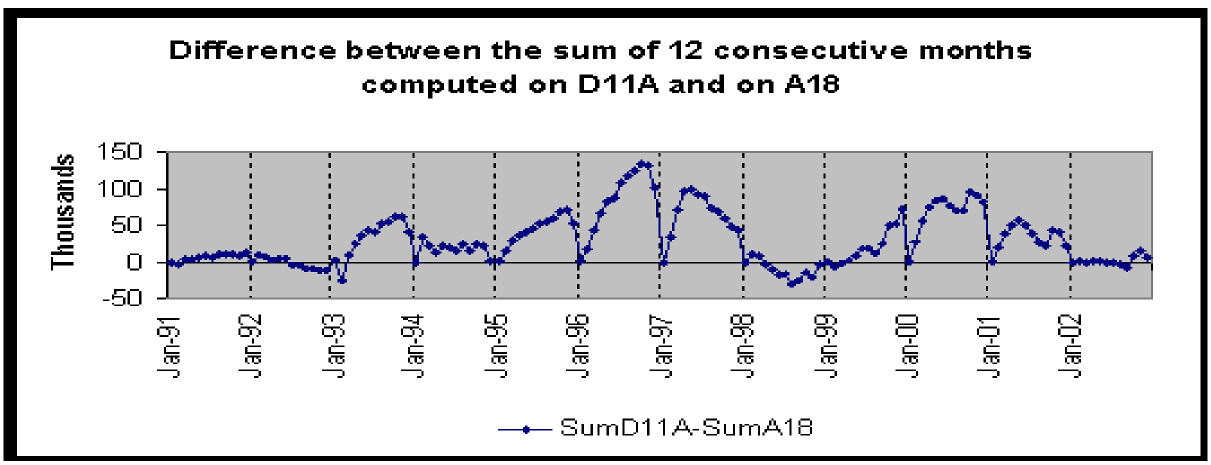

Figure 7: Canadian Department Stores Retail Trade Sales: monthly differences in the running 12-month sums in D11A and the calendar adjusted raw series (A18).