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REUSABLE COMPONENTS FOR BENCHMARKING USING KALMAN FILTERS

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INTRODUCTION

Kalman filters and smoothers are powerful algorithms that provide efficient solutions to many problems in the time series domain. This is certainly the case for some benchmarking approaches. We present in this paper how we have translated the versatility of the state-space forms and of the connected algorithms into software modules (called below "library"). For the time being, our library only tackles univariate state-space series. In spite of this important restriction, it can already be applied in many useful applications.

A large part of the work is devoted to the treatment of state-space forms in general. Special attention is paid to the initialisation of non stationary models, on the handling of the regression effects and on the definition of the likelihood. The theoretical developments follow the approach of Durbin and Koopman (2001, 2003).

As far as benchmarking is concerned, the present library concentrates on the estimation through single regression methods. It is much in the line of the work of Proietti (2004). That choice is mainly motivated by the wish to replace existing implementations of traditional algorithms. The new solution presents more alternatives (including the estimation of log-transformed models) and several diagnostics tools.

The work is on the frontier between recent statistical developments and software design; the structure of the paper reflects that duality. In the first paragraph, we briefly discuss the general technological choices necessary to ensure a high reusability. The second paragraph describes the theoretical framework; details on the different state-space representations and on the algorithms for filtering and smoothing are provided. Paragraph 3 outlines how the theory is translated into an extensible object-model. Finally two modules built on the library are briefly presented in the last paragraph.

The note is written as as technical documentation of the library for advanced users. In that perspective, most of the mathematical developments and of the discussion around the statistical methods for benchmarking have been deliberately omitted. Useful information on those topics can be found in many other papers.

The library and the different applications that use it can be freely dowloaded from a WEB server of the National Bank of Belgium (http://www.nbb.be/app/dqrd/index.htm).

1 TECHNOLOGY

The capability of reusing previous modules or of extending them to meet new situations constitutes a very important point in software development. This is of course necessary to be able to respond quickly to new needs. From the user's point of view, it is also a way to build a coherent framework for his work.

Statistical algorithms must often be integrated in completely different tasks. Benchmarking, for example, can be used in batch processing of many series; it can also be embedded in semi-automated tasks like the production of Quarterly National Accounts; advanced graphical interfaces should also be available for detailed analysis, while some facilities for more complex studies like Monte Carlo experiments are an interesting feature.

It is unlikely to find a single product that can respond efficiently to all those kinds of questions. However, as far as technology is concerned, object-oriented (OO) components based on standard technologies form a very interesting solution.

If their underlying technology is largely accepted by the software realm, OO components can quite easily be embedded in many different environments, from commercial software to a variety of tools

for in-house developments. Java is a popular solution when portability matters, while .NET becomes the norm for Windows applications. We provide implementations in both technologies, using the same object-model. It should also be stressed that, compared to other more traditional development languages like FORTRAN or C, those technologies yield much more robust solutions.

The OO paradigm is often limited to data encapsulation. While this feature is extremely useful for hiding the details of an implementation and for managing its complexity, the other characteristics of OO appear at least as important once extensibility is put forward.
Polymorphism allows the building of generic algorithms, based on general concepts rather than on specific cases, while inheritance allows specialisation or modification of existing implementations. These last two aspects of OO, and more specifically polymorphism permits a reasoning very close to the theoretical developments. They are widely exploited in the design of our framework on the state-space forms (SSF henceforth) and on the Kalman filters and smoothers (KF henceforth).

2 THEORETICAL FRAMEWORK

2.1 BASIC UNIVARIATE STATE SPACE MODEL

A detailed presentation of the theoretical framework on SSF is necessary not only to be able to interpret correctly the results and to compare them to other solutions, but also to understand the organization of the library, its limits as well as its capabilities of extension.
After the definition of the basic model, the formulae of the main KF algorithms are listed. We finally discuss how we deal with regression effects

2.1.1 DEFINITION

While the general linear gaussian state-space model can be written in many different ways, we shall use the presentation of Durbin and Koopman (DK henceforth, 2001), and we shall restrict it, as in the current library, to the univariate case.

Measurement equation:

$$y_t = Z_t \alpha_t + \varepsilon_t$$
$$\varepsilon_t \sim N\left(0, \sigma^2 h_t\right)$$

State equation:

$$\alpha_{t+1} = T_t \alpha_t + \eta_t$$
$$\eta_t \sim N\left(0, \sigma^2 V_t\right)$$

with $0 \le t < n$

y_t is the observation at period t, α_t is the r x 1 state vector. ε_t and η_t are assumed to be serially independent and independent of each other at all time points.

The residuals of the state equation will often be modelised as

$$\eta_t = RW\xi_t$$
$$\xi_t \sim N\left(0, \sigma^2 Q_t\right)$$

where ξ_t is a vector of e x 1 residuals (0 < e ≤ r), Q_t is a e x e covariance matrix, W_t is a m x e matrix (weights of the disturbances) and R_t is a r x m matrix composed of columns of I_r, that identifies the items of the state vector modified by the residuals.

(Diffuse) Initialisation

The initial conditions of the filter are defined as follows:

$$\alpha_0 = a_0 + \delta_0 + \eta_0$$
$$E(\delta_0) = 0$$
$$\mathrm{var}(\delta_0) = \kappa P_{\infty,0}$$
$$E(\eta_0) = 0$$
$$\mathrm{var}(\eta_0) = \sigma^2 P_{*,0}$$
$$\mathrm{cov}(\delta_0, \eta_0) = 0$$

where κ is arbitrary large[1].
P_* is the variance of the stationary elements of the initial state vector and κP_∞ models the diffuse part. Both matrices are r x r[2]. We also suppose that the rank of P_∞ is d.

In summary, the matrices of the system are:

Matrices	Dimensions	Meaning
y_t	scalar	Observation
α_t	r x 1	State vector
ε_t	scalar	disturbance of the observation equation
η_t	e x 1	disturbances of the state equation
h_t	scalar	variance of ε_t, apart from the σ^2 factor
Q_t	e x e	covariance of η_t, apart from the σ^2 factor (partim, see below)
W_t	m x e	part of the covariance of η_t (weights of disturbances)
R_t	r x m	part of the covariance of η_t (columns of identity)
V_t	r x r	The full covariance matrix of the disturbances of the state equation $(= R_t W_t Q_t W_t' R_t')$, apart from the σ^2 factor
S_t	r x e	$= R_t W_t Q_t$. Auxiliary matrix used for smoothing
Z_t	1 x r	measurement matrix
T_t	r x r	transition matrix
a_0	r x 1	initial state
$P_{*,0}$	r x r	covariance of the stationary part of the initial state, apart from the σ^2 factor
$P_{\infty,0}$	r x r	covariance (apart from a factor) of the diffuse part of the initial state

[1] The σ^2 factor is absorbed in κ

[2] We do not require that diffuse/non diffuse elements reside in separate items of the initial state vector. Diffuse/non diffuse effects must simply be splitsed in independent parts.

2.1.2 FILTERING, SMOOTHING AND LIKELIHOOD EVALUATION

State space models are efficiently treated by KF. The augmented KF of De Jong (91) and a variant due to Gomez and Maravall (93) can be used to provide an exact solution when the model contains elements with unspecified distributions. However, the approach of Durbin and Koopman, used in our modules, provides simpler and more efficient algorithms.

This paragraph recapitulates the recursions for filtering, (disturbance) smoothing and likelihood evaluation. Apart from some rearrangements, their mathematical derivations can be found in DK (2001, 2003).
Those small rearrangements pursue several goals: getting a faster processing through vector computation, ensuring the symmetry of covariance matrices and limiting the number of patterns in the derivations (see § 3.1.1 for a discussion on that point).
The library is a straightforward application of the formulae given below[3]:

2.1.2.1 Recursions for filtering

We use the following notations:

$$a_{t+1} = E\left(\alpha_{t+1} | y_0, ..., y_t\right)$$
$$P_{t+1} = \mathrm{var}\left(\alpha_{t+1} | y_0, ..., y_t\right)$$
$$v_t = y_t - E\left(y_t | y_0, ..., y_{t-1}\right) = y_t - Z_t a_t$$

Normal recursions

When y_t is *not missing* the normal recursions are

$$f_t = Z_t P_t Z_t' + h_t$$
$$v_t = y_t - Z_t a_t$$
$$M_t = P_t Z_t'$$
$$C_t = T_t M_t$$
$$a_{t+1} = T_t a_t + C_t v_t / f_t$$
$$P_{t+1} = T_t P_t T_t' - C_t C_t' / f_t + V_t$$

If y_t is missing, they are simplified to

$$a_{t+1} = T_t a_t$$
$$P_{t+1} = T_t P_t T_t' + V_t$$

Diffuse recursions

Diffuse recursion appends during the first periods; the length of the diffuse part is dynamically determined during the filtering process. When y_t is *not missing*, a first set of formulae is defined by

---

[3] σ^2 is set to 1 (except for the likelihood evaluation). <A> stands for A+A'.

$$f_{*,t} = Z_t P_{*,t} Z_t' + h_t$$
$$M_{*,t} = P_{*,t} Z_t'$$
$$C_{*,t} = T_t M_{*,t}$$
$$f_{\infty,t} = Z_t P_{\infty,t} Z_t'$$
$$M_{\infty,t} = P_{\infty,t} Z_t'$$
$$C_{\infty,t} = T_t M_{\infty,t}$$
$$v_t = y_t - Z_t a_t$$

Following the value of f_∞, two cases have then to be considered. When $f_\infty \neq 0$, we have

$$a_{t+1} = T_t a_t + v_t C_{\infty,t} / f_{\infty,t}$$
$$P_{\infty,t+1} = T_t P_{\infty,t} T_t' - C_{\infty,t} C_{\infty,t}' / f_{\infty,t}$$
$$P_{*,t+1} = T_t P_{*,t} T_t' + C_{\infty,t} C_{\infty,t}' f_{*,t} / f_{\infty,t}^2 - < C_{*,t} C_{\infty,t}' > / f_{\infty,t} + V$$

On the other hand, when $f_\infty = 0$,

$$a_{t+1} = T_t a_t + C_{*,t} v_t / f_{*,t}$$
$$P_{\infty,t+1} = T_t P_{\infty,t} T_t'$$
$$P_{*,t+1} = T_t P_{*,t} T_t' - C_{\infty,t} C_{\infty,t}' / f_{*,t} + V_t$$

When y_t is *missing*, the formulae are much simpler:

$$a_{t+1} = T_t a_t$$
$$P_{\infty,t+1} = T_t P_{\infty,t} T_t'$$
$$P_{*,t+1} = T_t P_{*,t} T_t' + V_t$$

The other quantities are not computed.

2.1.2.2 Recursions for (disturbance) smoothing

(n) indicates formulae for normal smoothing, (d) for disturbance smoothing, while common formulae are noted (c).

We use the following notations:

$$\widetilde{a}_t = E(\alpha_t | y_0, ..., y_n)$$
$$\widetilde{P}_t = \mathrm{var}(\alpha_t | y_0, ..., y_n)$$
$$u_t = E(\eta_t | y_0, ..., y_n)$$
$$e_t = E(\varepsilon_t | y_0, ..., y_n)$$

u_t is presented as a row-matrix. v_t, f_t, a_t, P_t, C_t, $f_{\infty,t}$, $f_{*,t}$, $P_{\infty,t}$, $P_{*,t}$, $C_{\infty,t}$, $C_{*,t}$ are quantities obtained in the filtering process.

r_t, $r_{0,t}$ and $r_{1,t}$ are auxiliary row-matrices, while N_t, $N_{0,t}$, $N_{1,t}$ and $N_{2,t}$ are auxiliary square matrices.
Except $r_{0,t}$ and $N_{0,t}$ which are initialised with the last values of r_t and N_t, those objects are set to 0 at the beginning of the process.
It is to be noted that the recursions on the variances (in brackets), which are by far the most time consuming, may be omitted.

Normal recursions

When y_t is *not missing*, the normal recursions are

$$\text{(c)}\quad \mathrm{K_t} = C_t / f_t$$
$$\text{(c)}\quad r_{t-1} = (v_t / f_t - r_t K_t) Z_t + r_t T_t$$
$$\text{(c)}\quad [L_t = T_t - K_t Z_t]$$
$$\text{(c)}\quad [N_{t-1} = Z_t' Z_t / f_t + L_t' N_t L_t]$$

The *smoothed states* and their variances are given by

$$\text{(n)}\quad \tilde{a}_t' = a_t' + r_t P_t$$
$$\text{(n)}\quad [\tilde{P}_t = P_t - P_t' N_t P_t]'$$

while the *smoothed disturbances* and their variances are expressed as

$$\text{(d)}\quad e_{\mathrm{t}} = (v_t / f_t - r_t K_t) h_t$$
$$\text{(d)}\quad u_{\mathrm{t}} = r_{\mathrm{t}} S_t$$
$$\text{(d)}\quad [\mathrm{var}(e_{\mathrm{t}}) = h_t - h_t^2 (1 / f_t + K_t' N_t K_t)]$$
$$\text{(d)}\quad [\mathrm{var}(u_{\mathrm{t}}) = Q_t - S_t' N_t S_t]$$

When y_t is *missing*, the recursions are reduced to

$$\text{(c)}\quad r_{t-1} = r_t T_t$$
$$\text{(c)}\quad [N_{t-1} = T_t' N_t T_t]'$$

the smoothed states are computed as above, and the smoothed disturbances are missing.

Diffuse recursions

For the initial time period, when y_t is *not missing*, we consider two cases as we did in the filtering. If $f_\infty \neq 0$, the recursion formulae are

(c) $\mathrm{K}_{\infty,\mathrm{t}} = C_{\infty,t} / f_{\infty,t}$

(c) $\mathrm{K}_{*,\mathrm{t}} = C_{*,t} / f_{\infty,t} - C_{\infty,t} f_{*,t} / f_{\infty,t}^2$

(c) $\mathrm{c}_{0,\mathrm{t}} = -r_{0,t} K_{\infty,\mathrm{t}}$

(c) $\mathrm{c}_{1,\mathrm{t}} = v_t / f_{\infty,t} - r_{0,t} K_{*,t} - r_{1,t} K_{\infty,t}$

(c) $r_{0,t-1} = c_{0,t} Z_t + r_{0,t} T_t$

(c) $r_{1,t-1} = c_{1,t} Z_t + r_{1,t} T_t$

(c) $\left[L_{\infty,t} = T_t - K_{\infty,t} Z_t \right]$

(c) $\left[N_{0,t-1} = L'_{\infty,t} N_{0,t} L_{\infty,t} \right]$

(c) $\left[N_{1,t-1} = Z'_t Z_t / f_{\infty,t} + L'_{\infty,t} N_{1,t} L_{\infty,t} - < Z'_t K'_{*,t} N_{0,t} L_{\infty,t} > \right]$

(c) $\left[N_{2,t-1} = Z'_t Z_t \left(K'_{*,t} N_{0,t} K_{*,t} - f_{*,t} / f_{\infty,t}{}^2 \right) + L'_{\infty,t} N_{1,t} L_{\infty,t} - < Z'_t K'_{*,t} N_{1,t} L_{\infty,t} > \right]$

The *smoothed states* and their variances are given by

(n) $\widetilde{a}'_t = a'_t + r_{0,t} P_{*,t} + r_{1,t} P_{\infty,t}$

(n) $\left[\widetilde{P}_t = P_{*,t} - P'_{*,t} N_{*,t} P_{*,t} - < P'_{*,t} N_{1,t} P_{\infty,t} > - P'_{\infty,t} N_{2,t} P_{\infty,t} \right]$

The *smoothed disturbances* and their variances are expressed as

(d) $e_{\mathrm{t}} = -c_{0,t} h_t$

(d) $u_{\mathrm{t}} = r_{\mathrm{t}} S_t$

(d) $\left[\mathrm{var}(e_{\mathrm{t}}) = h_t - h_t^2 K'_{\infty,t} N_{0,t} K_{\infty,t} \right]$

(d) $\left[\mathrm{var}(u_{\mathrm{t}}) = Q_t - S'_t N_{0,t} S_t \right]$

When $f_\infty = 0$, the formulae become

(c) $\mathrm{K}_{*,\mathrm{t}} = C_{*,t} / f_{*,t}$

(c) $\left[L_{*,t} = T_t - K_{*,t} Z_t \right]$

(c) $r_{0,t-1} = \left(v_t / f_{*,t} - r_{0,t} K_{*,t} \right) Z_t + r_{0,t} T_t$

(c) $r_{1,t-1} = r_{1,t} T_t$

(c) $\left[N_{0,t-1} = Z'_t Z_t / f_{*,t} + L'_{*,t} N_{0,t} L_{*,t} \right]$

(c) $\left[N_{1,t-1} = T'_t N_{1,t} L_{*,t} \right]$

(c) $\left[N_{2,t-1} = T'_t N_{2,t} T_t > \right]$

The *smoothed states* and their variances are derived as in the previous case, while the s*moothed disturbances* and their variances are given by formulae similar to those of the normal case

(d) $e_{\mathrm{t}} = \left(v_t / f_{*,t} - r_t K_{*,t} \right) h_t$

(d) $u_{\mathrm{t}} = r_{\mathrm{t}} S_t$

(d) $\left[\mathrm{var}(e_{\mathrm{t}}) = h_t - h_t^2 \left(1 / f_{*,t} + K'_{*,t} N_{0,t} K_{*,t} \right) \right]$

(d) $\left[\mathrm{var}(u_{\mathrm{t}}) = Q_t - S'_t N_{0,t} S_t \right]$

When y_t is *missing*, the recursions are reduced to

$$\text{(c)} \quad r_{0,t-1} = r_{0,t} T_t$$
$$\text{(c)} \quad r_{1,t-1} = r_{1,t} T_t$$
$$\text{(c)} \quad \left[N_{0,t-1} = T_t' N_{0,t} T_t \right]$$
$$\text{(c)} \quad \left[N_{1,t-1} = T_t' N_{1,t} T_t \right]$$
$$\text{(c)} \quad \left[N_{2,t-1} = T_t' N_{2,t} T_t \right]$$

The *smoothed states* are then computed as above, while the s*moothed disturbances* are missing.

As usual, a fast estimation of the smoothed states is easily obtained from the smoothed disturbances[4]:

$$\tilde{a}_{t+1} = T_t \tilde{a}_t \left[+ S_t u_t' \right]$$

with the initialisation

$$\tilde{a}_0' = a_0' + r_{0,0} P_{*,0} + r_{1,0} P_{\infty,0} .$$

This last procedure cannot be applied when an estimation of the variances is needed.

2.1.3 LIKELIHOOD EVALUATION

Skipping items corresponding to missing values, the likelihood of the model (1) is easily obtained by means of the so-called prediction error decomposition :

$$\ln L_d(y) = -\frac{1}{2} \left\{ n \ln 2\pi + \sum_{t=q}^{t<n} v_t^2 / (f_t \sigma^2) + \sum_{t=q}^{t<n} \ln\left(\sigma^2 f_t\right) + \sum_{t=0}^{t<q} w_t \right\}$$

where

$w_t = \ln\left(f_{\infty,t}\right)$ if $f_{\infty,t} \neq 0$, $w_t = v_t^2 / (f_{*,t} \sigma^2) + \ln\left(f_{*,t}\right)$ otherwise.

To simplify the notation, we shall suppose below that the d indexes for which $f_{\infty,t} \neq 0$ are at the front and we shall assimilate the other diffuse items to the non diffuse part.

Maximizing the likelihood for σ^2, we get

$$\ln L_d(y) = -\frac{1}{2} \left\{ n \ln 2\pi + (n-d)\left(\ln \hat{\sigma}^2 + 1\right) + \sum_{t=d}^{t<n} \ln f_t + \sum_{t=0}^{t<d} \ln f_{\infty,t} \right\}$$
$$\hat{\sigma}^2 = \left(\sum_{t=d}^{t<n} v_t^2 / f_t \right) / (n-d)$$

2.1.4 REGRESSORS EFFECTS

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[4] $S_t u_t'$ is dropped when the smoothed disturbances are missing.

We consider in this paragraph a measurement equation enriched by regressors:

$$y_t = Z_t\alpha_t + X_t\beta + \varepsilon_t \text{, (2)}$$

where X_t is the 1 x b row-matrix of regressors at time t, and β is the b x 1 column-matrix of coefficients. Other parts of the model remain unchanged.

We don't deal explicitly with regressors effects in the state equation. However such effects may always be moved in the the measurement equation by properly modifying them.
The β can be viewed as fixed but unknown (Rosenberg, 1973) or as diffused (De Jong, 1991). Unlike other potential undefined items of the initial state, which we always treat as diffuse, both cases are handled.

DK propose two solutions for the estimation of that model (2001, § 6.2.2 and 6.2.3): by extending the state vector with the coefficients of the regressors (extended model) or by using an approach similar to the augmented KF (augmented model). Our library provides implementations for both solutions.

2.1.4.1 Extended model[5]

The state-space model of the extended model is written as follows:

$$y_t = \tilde{Z}_t\tilde{\alpha}_t + \varepsilon_t$$
$$\tilde{\alpha}_{t+1} = \tilde{T}_t\tilde{\alpha}_t + \tilde{\eta}_t$$

where

$$\tilde{\alpha}_t = \begin{pmatrix} \alpha_t \\ \beta \end{pmatrix}$$
$$\tilde{Z}_t = \begin{pmatrix} 1 & X_t \end{pmatrix}$$
$$\tilde{T}_t = \begin{pmatrix} T_t & 0 \\ 0 & I_b \end{pmatrix},$$
$$\tilde{\eta}_t = \begin{pmatrix} \eta_t \\ 0 \end{pmatrix}$$

with the initial conditions

$$\tilde{P}_{*,0} = \begin{pmatrix} P_{*,0} & 0 \\ 0 & 0 \end{pmatrix}$$
$$\tilde{P}_{\infty,0} = \begin{pmatrix} P_{\infty,0} & 0 \\ 0 & I_b \end{pmatrix}.$$
$$\tilde{\alpha}_0 = \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix}$$

---

[5] The extended model should be used only in the case of diffuse regression effects. In the fixed case, we only have to replace y_t by $y_t - X_t\overline{\beta}$, with $\overline{\beta}$ estimated otherwise.

This model can be handled exactly as any other model of type (1). Thus, usual algorithms for filtering, smoothing and likelihood evaluation remain valid.

2.1.4.2 Augmented model

Estimation by means of the augmented approach consists essentially of two stages.

- In a first step, the KF is applied on y_t and on each regressors; the standardized residuals are obtained.
- The second step is simply an estimation of the OLS problem between the filtered series. De Jong (1991) solves it through normal equations, while Gomez-Maravall (1993) propose the use of the QR algorithm. The former better fits the case of models regularly updated with new observations, while the latter provides a more numerically stable solution.

Our implementation follows the QR solution. It is formalized below:

1. Filtering y_t with the DK algorithm. The n-d standardized residuals $v_{y,t}$ are stored[6].
2. Filtering of each regressor with the DK algorithm, reusing the gain matrices calculated in 1. The b arrays of (n-d) standardized residuals $v_{x_i,t}$ are stored.
3. Solving the OLS problem $v_{y,t} = v_{X,t}\beta + \zeta_t$ by the QR algorithm; it provides $\hat{\beta}$, its covariance matrix ($= (UU')^{-1}$, where U is the upper triangular matrix in the R matrix of the decomposition), and a set of (n-d-b) independent residuals.

Under the fixed unknown assumption, the profile likelihood equals:

$$\ln L_c(y) = -\frac{1}{2}\left\{ n\ln 2\pi + (n-d)\left(\ln \hat{\sigma}^2 + 1\right) + \sum_{t=d}^{t<n} \ln f_t + \sum_{t=0}^{t<d} \ln f_{\infty,t} \right\}$$

$$\hat{\sigma}^2 = \left(\sum_{t=d+b}^{t<n} e_t^{\,2} \right) / (n-d)$$

where e_t are the (n-d-b) independent residuals obtained from the QR decomposition.

Under the diffuse assumption, the profile likelihood becomes:

$$\ln L_d(y) = -\frac{1}{2}\left\{ n\ln 2\pi + (n-d-b)\left(\ln \hat{\sigma}^2 + 1\right) + \sum_{t=d}^{t<n} \ln f_t + \sum_{t=0}^{t<d} \ln f_{\infty,t} + \ln\left|UU'\right| \right\}$$

$$\hat{\sigma}^2 = \left(\sum_{t=d+b}^{t<n} e_t^{\,2} \right) / (n-d-b)$$

where e_t is as above. As U is a triangular matrix, its determinant is trivially obtained.

This process is equivalent to the estimation through the augmented KF with collapsing followed by a QR decomposition, as developed by Gomez and Maravall and implemented in the software TRAMO. However, the initialisation problem is tackled here by means of the DK algorithm.

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[6] They correspond to the indexes where $f_\infty = 0$.

The extended approach and the augmented approach yield the same likelihood evaluation. Both solution present some advantages. The former produces recursive residuals and recursive estimates of the coefficients. Recursive residuals are serially uncorrelated and easily interpretable, while recursive estimates are interesting for a study on the revisions. The latter is more stable and often faster.
We use the extended model in the analysis stage while we prefer the augmented model during the optimization procedure.

2.2 BENCHMARKING

2.2.1 GENERALITIES

The main goal of the present implementation of benchmarking is to provide a solution that encompasses popular techniques (Chow-Lin, Fernandez, Litterman, Denton, ...)
Like Proietti (2004) we concentrate on single regression equations, with errors described by state-space forms.
The library gives solutions for the distribution of a flux variable as well as for the interpolation of a stock variable. Estimation of the latter doesn't require special devices; it just amounts to the familiar problem of missing observations. We shall not consider it in details. The distribution case is handled through the use of cumulator variables (see Harvey, 1989, § 6.3).
We shall also discuss the problem of log-transformed models, for which we provide approximated and iterative solutions. This last point is once again largely inspired by the work of Proietti (2004).

2.2.2 DISAGGREGATED TIME SERIES MODEL

We suppose that the disaggregated time series admits a state space representation similar to (2). However, to simplify the derivation, we do not allow a disturbance term in the measurement equation[7].

In summary, the disaggregated model is

$$y_t = \mu_t [+ X_t \beta]$$
$$\mu_t = Z_t \alpha_t$$
$$\alpha_{t+1} = T_t \alpha_t + \eta_t$$

with the other hypotheses as above.

Apart from some assumptions on the initial values, the most popular disaggregation methods fit this form (see Proietti (2004) for a detailed discussion on that point); this is summarized in the following table:

Methods	*Regressors*	*Residuals (μ_t)*
Chow-Lin	constant + related series	ARIMA(1, 0, 0)
Litterman	[linear trend +] related series	ARIMA(1, 1, 0)
Fernandez	[linear trend +] related series	ARIMA(0, 1, 0)
Denton (K)		ARIMA(0, K, 0)

[7] We can easily get around that condition by adding the disturbance term, if any, to the state vector.

2.2.3 AGGREGATION

2.2.3.1 State-space form

In the distribution problem, y_t is not available. However, the temporally aggregated series is observed at a lower frequency. If we note q the number of high-frequency periods in one low-frequency period, the aggregated series can be written[8]:

$$Y_t = \sum_{j=0}^{j<q} y_{qt+j}$$

For any series z_t, we define the auxiliary cumulator variable z_t^C. If $[t]_q$ refers to the highest multiple of q lower or equal to t,

$$z_t^C = \sum_{j=[t]_q}^{t} z_t$$

The cumulated series y_t^C is equal to Y_t when $t+1$ is a multiple of q and undefined elsewhere.
We also define:

$$z_t^{\underline{C}} = \sum_{j=[t]_q}^{t-1} z_t$$

Thus, $z_t^C = z_t^{\underline{C}} + z_t$ and $z_t^{\underline{C}} = 0$ if t is a multiple of q.

Seeing that the operator $(.)^C$ is linear, we can use the following state space representation for the cumulated series:

$$\begin{aligned} y_t^C &= \tilde{\mu}_t^C [+ X_t^C \beta] \\ \tilde{\mu}_t^C &= \tilde{Z}_t \tilde{\alpha}_t \qquad (3) \\ \tilde{\alpha}_{t+1} &= \tilde{T}_t \tilde{\alpha}_t + \tilde{\eta}_t \end{aligned}$$

where

$$\tilde{\alpha}_t = \begin{pmatrix} \mu_t^{\underline{C}} \\ \alpha_t \end{pmatrix}$$

$$\tilde{Z}_t = \begin{pmatrix} 1 & Z_t \end{pmatrix}$$

$$\tilde{T}_t = \begin{pmatrix} \delta_t & \delta_t Z_t \\ 0 & T_t \end{pmatrix}$$

---

[8] Like in the programming languages used in our developments (C++, Java, C#), we shall always consider 0-based indexes: y(0) is the first observation of y, the place of the first month in the year is 0, ...

$$\tilde{\eta}_t = \begin{pmatrix} 0 \\ \eta_t \end{pmatrix}$$

$$\tilde{\alpha}_0 = \begin{pmatrix} 0 \\ \alpha_0 \end{pmatrix}$$

with $\delta_t = 0$ if $t+1$ is a multiple of q, $\delta_t = 1$ otherwise.

It is to be noted that the disaggregated series can be easily retrieved from the state vector of model (3) by means of the matrix $Z_{\mu_t} = \begin{pmatrix} 0 & Z_t \end{pmatrix}$; we have indeed $y_t = Z_{\mu_t}\alpha_t \left[+ X_t\beta\right]$.

2.2.3.2 Estimation of the aggregated model

The estimation of the aggregated model by itself is a straighforward application of the KF on the "cumulated" model (3).

As far as performances matter, the computing of the smoothed estimate $\tilde{y}_t = E(y_t|n)$ of the disaggregated series requires some care. Depending on the presence of regressors and wether we need to obtain the variance of the smoothed series or not, we use the following schemes:

- No regressors, disaggregated series only
  - Disturbance smoother on y_t^C
  - $\tilde{y}_t = Z_{\mu_t}\tilde{\alpha}_t$
- No regressors, disaggregated series and its variance
  - Standard smoother on y_t^C
  - $\tilde{y}_t = Z_{\mu_t}\tilde{\alpha}_t$
  - $\mathrm{var}(\tilde{y}_t) = Z_{\mu_t}\,\mathrm{var}(\tilde{\alpha}_t)Z'_{\mu_t}$
- Regressors, disaggregated series only
  - Disturbance smoother on $y_t^C - X_t^C\hat{\beta}$.
  - $\tilde{v}_t = Z_{\mu_t}\tilde{\alpha}_t$
  - $\tilde{y}_t = \tilde{v}_t + X_t\hat{\beta}$
- Regressors, disaggregated series and its variance
  - Standard smoother on $y_t^C - X_t^C\hat{\beta}$.
  - $\tilde{v}_t = Z_{\mu_t}\tilde{\alpha}_t$
  - $\mathrm{var}(\tilde{v}_t) = Z_{\mu_t}\,\mathrm{var}(\tilde{\alpha}_t)Z'_{\mu_t}$
  - Disturbance smoother on each cumulated regressor. $\tilde{X}_t$ are obtained
  - $\tilde{y}_t = \tilde{v}_t + X_t\hat{\beta}$

- $\text{var}(\tilde{y}_t) = \text{var}(\tilde{v}_t) + (X_t - \tilde{X}_t)\,\text{var}(\hat{\beta})(X_t - \tilde{X}_t)'$

2.2.3.3 Logarithmic transformation

Time series are often modelled in terms of the logarithms of their original data. As the logarithmic transformation is not additive, the distribution method exposed above can not be applied directly. We shall consider an approximated solution, which sum up to a small adaptation of the previous method (see Di Fonzo, 2003) and the Iterative method developped by Proietti (2004). The former will be our starting point for the latter.

More formally, we consider in this paragraph that the state-space form is defined on the logarithmic transformation of the disaggregated series.

$$\begin{aligned} \ln y_t = z_t &= \mu_t [+ X_t \beta] \\ \mu_t &= Z_t \alpha_t \qquad (4) \\ \alpha_{t+1} &= T_t \alpha_t + \eta_t \end{aligned}$$

The aggregation constraint is now non linear. The different options implemented in the library are detailed below.

2.2.3.3.1 Approximated solution

By using the Taylor series expansion of $\ln y_t$ around the average of y_t during each aggregated period, it can be easily seen that

$$q \ln Y_t - q \ln q \approx \sum \ln y_q$$

The logarithmic model can thus be approximated by the linear solution, if we properly transform the aggregated values.

As suggested in Di Fonzo (2003), we shall restore the exact aggregation constraints by distributing the residuals with the Denton's method.

2.2.3.3.2 Iterative solution

Using a reasoning similar to the Proietti's one (2004), given a trial disaggregated series $\tilde{y}_t = \exp(\tilde{z}_t)$, the Taylor expansion of $y_t = \exp(z_t)$ around $\tilde{z}_t$ yields:

$$y_t \approx \exp(\tilde{z}_t)(1 - \tilde{z}_t) + \exp(\tilde{z}_t) z_t$$

Applying the cumulator operator, we get:

$$y_t^C - (\exp(\tilde{z}_t)(1 - \tilde{z}_t))^C = (\exp(\tilde{z}_t)\mu_t)^C [+(\exp(\tilde{z}_t) X_t)^C \beta]$$

If we pose

$$K_t = y_t^C - \left(\exp(\tilde{z}_t)(1-\tilde{z}_t)\right)^C$$
$$s_t = \exp(\tilde{z}_t)$$

the previous equation can be more concisely expressed as

$$K_t = \left(s_t \mu_t\right)^C [+\left(s_t X_t\right)^C \beta]$$

The state-space form of the linear gaussian approximated model (LGAM) conditional to $\tilde{z}_t$ is then formulated as follows:

$$\begin{aligned} (s_t \tilde{\mu}_t)^C &= \tilde{Z}_t \tilde{\alpha}_t \\ \tilde{\alpha}_{t+1} &= \tilde{T}_t \tilde{\alpha}_t + \tilde{\eta}_t \end{aligned} \quad (5)$$

with

$$\tilde{\alpha}_t = \begin{pmatrix} \left(s_t \mu_t\right)^C \\ \alpha_t \end{pmatrix}$$
$$\tilde{Z}_t = \begin{pmatrix} 1 & s_t Z_t \end{pmatrix}$$
$$\tilde{T}_t = \begin{pmatrix} \delta_t & \delta_t s_t Z_t \\ 0 & T_t \end{pmatrix}$$
$$\tilde{\eta}_t = \begin{pmatrix} 0 \\ \eta_t \end{pmatrix}$$
$$\tilde{\alpha}_0 = \begin{pmatrix} 0 \\ \alpha_0 \end{pmatrix}$$

Applying the KF and the disturbance smoother (see 4.4), a new estimate of the disaggregated series is computed. The process is iterated until convergence. As starting values, we take the solution of the approximated solution (without the final Denton correction) as defined in 2.2.3.3.1.
The likelihood of the model is approximated by that of the LGAM at convergence.
It should be noted that the iterative process doesn't necessarily converge. This is more often the case when the starting values are far from their optimum. If some parameters of the model have to be estimated by ML, a good solution seems to start the maximization procedure from the ML estimates of the approximated logarithmic model. Our implementation works in that way.

2.2.4 PRACTICAL CONSIDERATIONS

Aggregation leads to a substantial loss of information. In practice, only very parsimonious models can be estimated in a relatively stable way. Moreover, many SSF, for example those including seasonal unit roots, lead to degenerate aggregation models. Even if the library allows the use of complex SSF for the residuals, those simple observations justify why we shall restrict our applications, in many cases, to a small subset of models, including the most traditional solutions.

It should also be noted that, as far as ARIMA residuals are considered, models for distribution can be easily transformed to models for interpolation[9]: seeing that stocks are simply an accumulation of flux, we can move from one case to the other by accumulating the aggregated series and the

[9] This is not true when we are dealing with log-transformed models.

related series (the starting values don't matter) and by adding a unit root to the model of the residuals. The versatility of the library allows the statistical treatment and comparison of both solutions. This is also a way to verify some results with other software (for instance TRAMO).

3 OO-DESIGN

The object-oriented paradigm offers well-known facilities to achieve extensibility. We shall show in the next paragraphs how they are exploited in our library leading to efficient and flexible implementations of KF.

3.1 STATE SPACE MODELS

3.1.1 GENERAL CONSIDERATIONS

While general forms of KF, based on matrix computation, can be quite easily developed, faster implementations must exploit the structure of the matrices involved in the KF. Those structures are specific to each kind of model. By exploiting them, we can achieve a substantial gain in performances.
The OO-design of the library is based on those considerations. Common features, like the whole logic in the filtering, smoothing, likelihood evaluation, ... will be handled by general entities while the actual computation will be delegated to implementations of specific models.

We shall clarify that point by an example.

The filtering process involves several operations that multiply an array by the transformation matrix T_t (see 2.1.2, $K_t = T_t P_t Z_t' / f_t$, $a_{t+1} = T_t a_t + K_t v_t$).

Instead of using actual matrix computations, we shall impose on each specific SSF to provide a function that performs that multiplication ($y = T_t x = f_{Tx}(t,x)$). In many cases, that function will be simple and much faster than matrix computation.

For example, in the case of an ARIMA model (see appendix 1 for its SSF), the operation $y = f_{Tx}(t,x)$ is defined by

$$y(i-1) = x(i), 0 < i < r$$
$$y(r-1) = -\sum_{i=1}^{P} \phi(i) x(r-i)$$

where the $\phi(i)$ are the P coefficients of the autoregressive polynomial.

It involves P (<=r) multiplications while the blind matrix computation needs r^2 multiplications and more memory traffic.

A similar reasoning could be applied to other operations like, for example, $Z_t x = f_{Zx}(t,x)$ (multiplication by the measurement matrix), or like $xT_t = f_{xT}(t,x)$ (right-multiplication by the transition matrix, used in the smoothing process).

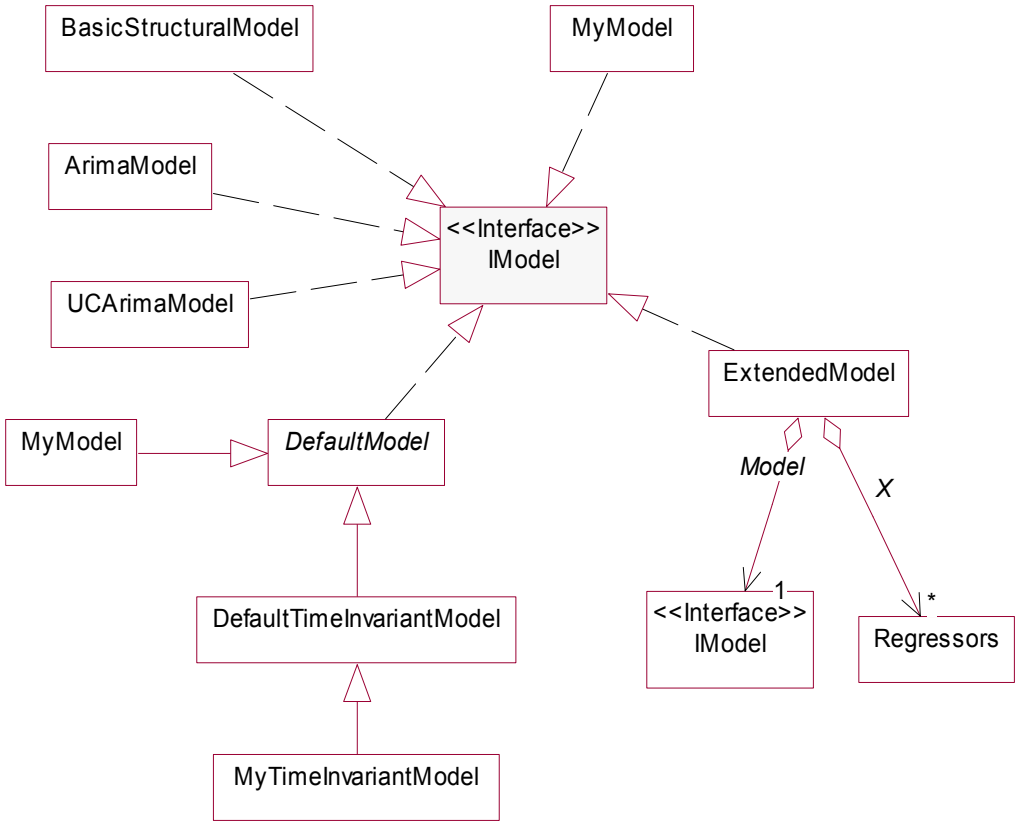
Thus, besides the description of the different matrices listed in § 2.1.1, the definition of a generic SSF (see appendix 2) also precise some "basic operations" for a faster processing. The common algorithms only call generic properties and methods; so they remain valid for any (new)

implementations of SSF. However, by using the "basic operations", they implicitely exploit the specific structure of each model.

The skeleton of our OO-model on SSF is illustrated in the following paragraphs by simplified UML diagrams.

3.1.2 MODELS HIERARCHY

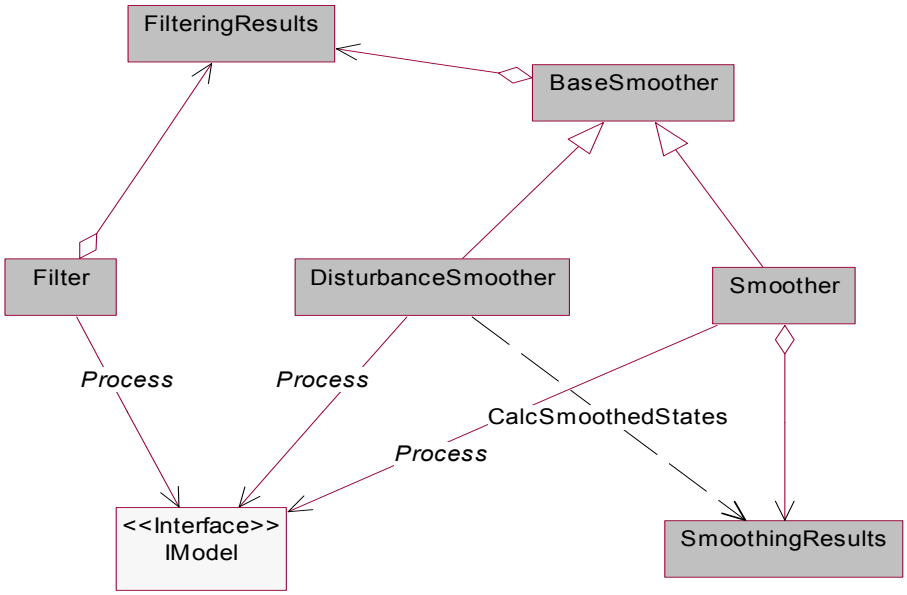
Any SSF is an implementation of the generic interface *IModel*. While some implementations are provided in the library (*BasicStructuralModel, ArimaModel, UCArimaModel*, ...), new ones can be added by the user, from scratch or by modifying existing solutions. For a "quick and dirty" development, the library also provides default implementations (*DefaultModel, DefaultTimeInvariantModel*); because they hardly exploit the structure of the model, the default implementations are significantly slower[10]. Finally, the *ExtendedModel (2.1.4.1)*, defined for any SSF, is simply another kind of model



3.1.3 FILTERING AND SMOOTHING

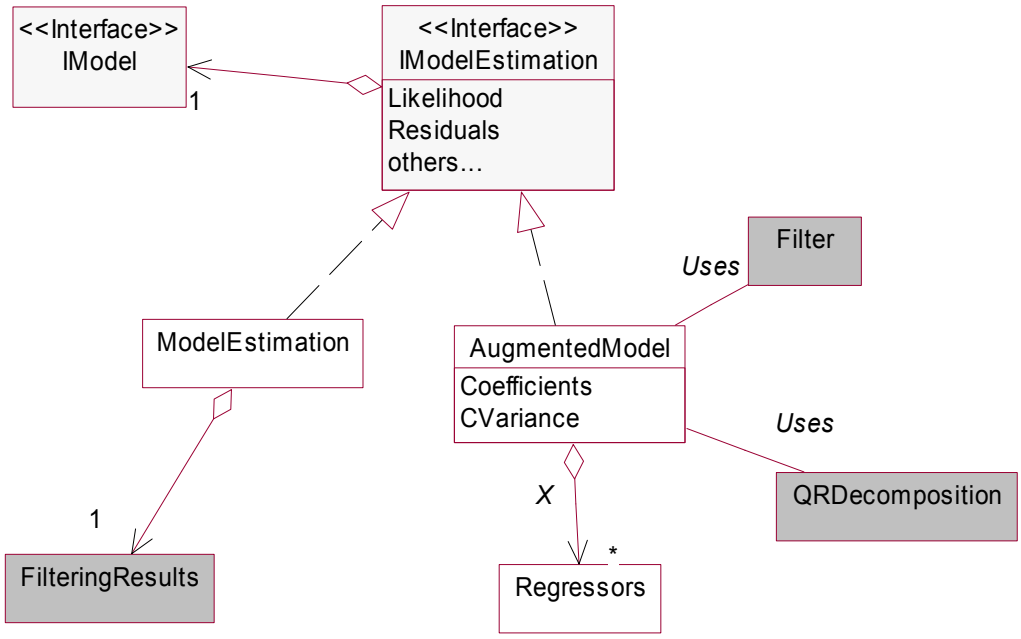
Filtering and (disturbance)smoothing are monitored by special entities. Those processes are defined for any SSF. Results are stored in independent objects for possible future uses.

[10] In general, between two and three times slower than optimized implementations.



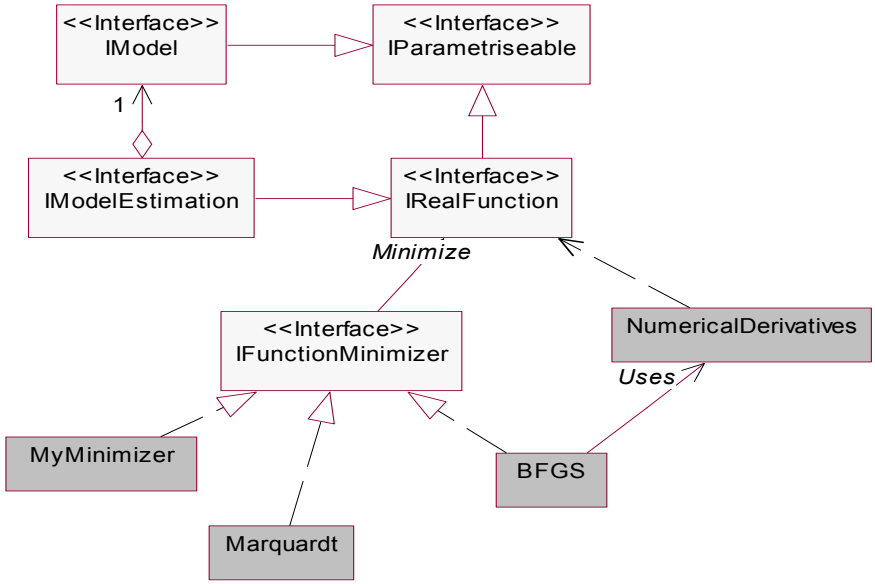
3.1.4 ESTIMATION

Estimation of any SSF is handled by the *AugmentedModel* (2.1.4.2) or by the *ModelEstimation* (2.1.3) entities depending on whether regressors are present or not. Likelihood, residuals, coefficients of the regression and other results are provided by those entities.



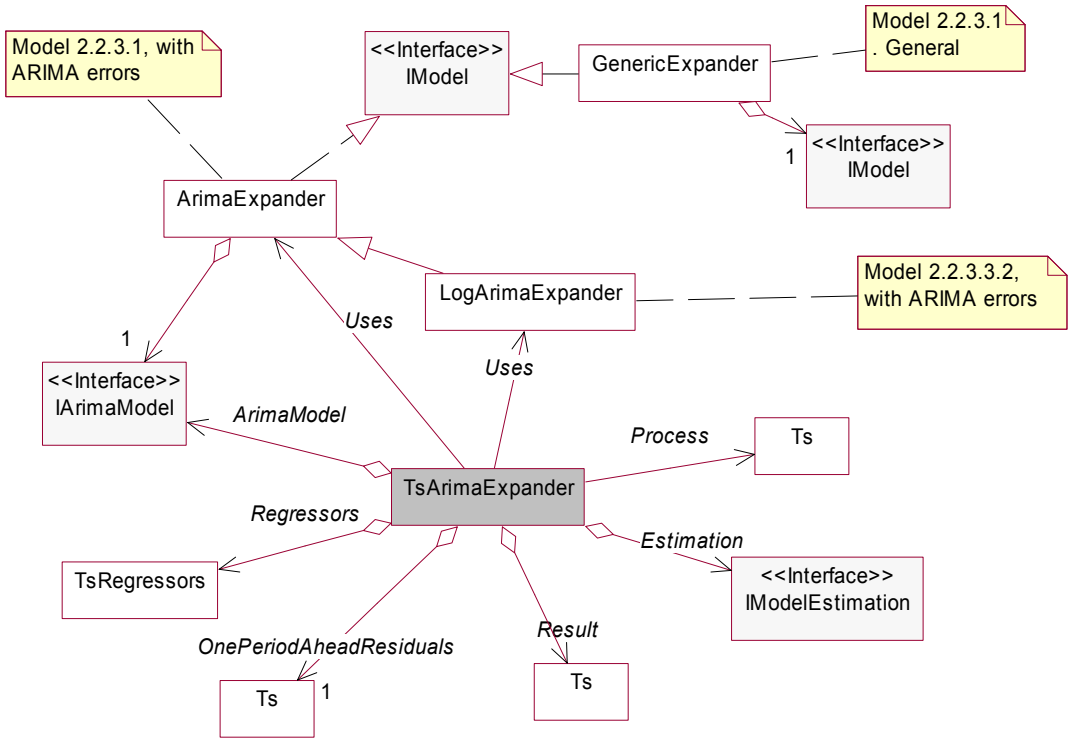
3.1.5 LIKELIHOOD MAXIMISATION

An *IModelEstimation* can be viewed as a real function; its parameters are the hyperparameters of the underlying SSF. Any function minimizer can then be used to search the maximum likelihood estimates of those parameters. In most cases, the BFGS algorithm provided in the library, which is based on numerical derivatives, yields good results.



3.2 BENCHMARKING

Apart from some small details, our implementation of benchmarking is a straighforward application of the KF algorithms of § 2.1, with the different SSF defined in § 2.2.
The library provides three kinds of benchmarking models. The first one, *GenericExpander*, is an exact translation of the model (3): it accepts residuals represented by any SSF, provided that its measurement equation doesn't contain disturbances. A*rimaExpander* objects are entities specialized for the handling of benchmarking models with ARIMA residuals; the most popular solutions belong to that case. Finally, the *LogArimaExpander* class modifies the previous one to deal with the state-space form (5), used in the iterative approach of the log-transformed model.



The high-level entity *TsArimaExpander* allows an easier treatment of the different solutions in the case of SSF with ARIMA residuals. It also provides many auxiliary results and hides most of the

details involved in a complete processing. It is also the cornerstone of the applications presented in §4.

4 APPLICATIONS

The basic idea behind our technological choice is to provide a reusable toolbox rather than a closed end product. As shown in appendix 3, OO languages lead to code that doesn't go stray too far from the theoretical reasoning; so they facilitate fast and understandable developments .
However, to illustrate what can be done with the library, we provide two applications on benchmarking. The first one is a standard stand-alone application (NbbBenchmarking), with rich graphical possibilities, while the second one is an Excel add-in (Desaggregator.xls). Both allow the treatment of models that belong to the class defined in 2.2, when the residuals follow a simple ARIMA process. The possible specifications and the main results are detailed below[11]. More advanced documentation is available on the site.

Available options (specifications)

Arima model of the residuals
- AR(1): Chow-Lin
- RW1 (=ARIMA(0 1 0)): Denton(1), Fernandez
- RWAR1 (=ARIMA(1 1 0)): Litterman
- RW2 (=ARIMA(0 2 0)): Denton(2)
- SAR1 (=SARIMA(1 0 0)(1 0 0)): for seasonal residuals
- IMA(1, 1) (=reduced form of local level)
- IMA(2, 2) (=reduced form of local linear trend)

Aggregation type:
- Sum (flux)
- Average (flux)
- Last (stock)
- First (stock)

Objective function
- Likelihood
- Sum of the square residuals
- None (fixed parameter(s))

Regressors (see 2.1.4)
- Fixed unknown
- Diffuse

Log Transformation
- None
- Approximated (2.2.3.3.1)
- Exact (iterative, 2.2.3.3.2)

Trend and constant : admissible choices:

	Constant	**Trend**
AR1	V	V
RW		V
RWAR1		V
RW2		

[11] Features marked by an asterisk are only available in the stand-alone application.

SAR1	V	V
Local level		V
Local linear trend		

Trend and constant are defined for the measurement equation only.

Output

disaggregated series an its variance

Main Results

- Used specifications
- Estimated Arima model :(full) autoregressive and moving average polynomials
- Model adequation (computed on the extended model)
    - Log likelihood (3.2.3)
    - Prediction error variance (computed at the last non missing observation)
    - Standard error (ML estimator)
    - AIC
    - BIC
- Regressors
    - Estimates, SER and P-Value
    - Covariance matrix

Residuals analysis

The analysed residuals are the one-step-ahead forecast errors. They are computed by means of the extended KF (3.3.1)

- Statistics
    - Distribution
        - Mean
        - Skewness
        - Kurtosis
        - Join test (Doornik-Hansen)
    - Independence
        - Durbin-Watson
        - Ljung-Box
    - Randomness
        - Runs around the mean (number and length)
        - Up and down runs (number and length)
    - Linearity
        - Ljung-Box on square residuals
- Data* (graphical)
- (Partial)autocorrelations* (graphical)
- Periodogram* (graphical)
- histogram of the distribution* (graphical)

*Likelihood**

Profile likelihood function (maximised according to β, σ^2). 3D view in the case of 2 hyperparameters.

*Revisions history**

The way the revision history is computed can be customized as follow:

- The hyperparameters can be fixed or not (in the former case, the parameters computed on the full period are kept, but the other quantities are re-estimated, while in the latter case, the full process is reexecuted on different time horizons).
- The number of re-estimations (with a minimun of observations) involved in the study may be defined.

The results are (if q is the number of high-frequency periods in one low-frequency period):

- Errors

  we consider the absolute differences between the final estimated values (on the whole period) and the results obtained on shorter periods. The errors are computed on the last q periods of the estimation and on the first q forecasting periods.

  - mean of the errors
  - mean of the percentage errors
  - root mean square errors

  The other results come from the estimation of the SSF on shortened periods.

- Regressors: coefficients of the regression, with their standard deviations
- Estimates: interpolated series (with q forecasts)
- Parameters

Algorithms

We summarize below the procedures used according to the kind of aggregation, the presence of a regression part in the model and the log transformation option. The links with the theoretical part of the paper are highlighted.

		Sum/average (distribution of flux)	**Last/first observation (interpolation of stocks)**
No regressors	**No Log**	Normal estimation (2.1.3) of the cumulated model (y_t^C).	Normal estimation (2.1.3) with y_t extended by missing values;
	Appr. Log	Normal estimation (2.1.3) of the corrected cumulated model ($q\ln y_t^C - q\ln q$ or $\ln y_t^C$).	Normal estimation (2.1.3) with $\ln(y_t)$ extended by missing values;
	Iterative Log	Iterative estimation of the Log model (4.3.3.2), using the normal likelihood estimation (3.2.3).	
Regressors	**No Log**	Augmented/Extended model (2.1.4) on y_t^C and X_t^C. See 2.2.3.2 also for details.	Augmented/Extended model (2.1.4) with y_t extended by missing values;
	Appr. Log	Augmented/Extended model (2.1.4) of the corrected cumulated model ($q\ln y_t^C - q\ln q$ or $\ln y_t^C$), using X_t^C. See also 2.2.3.2 for details.	Augmented/Extended model (2.1.4) with $\ln(y_t)$ extended by missing values;

	Iterative Log	Augmented/Extended model (2.1.4) of the LGAM (2.2.3.3.2, iterative solution). See also 2.2.3.2 for other details.	

Maximum likelihood estimates are computed through the BFGS procedure integrated in the library. They depend on the assumption made on the diffuse/non diffuse character of the regression effects.

5 FUTURE EXTENSIONS

It should be stressed that the solutions for benchmarking proposed in the present library are just simple applications of the whole underlying SSF framework. Starting from the current situation, other approaches could be quickly implemented (for example the model of Durbin and Quenneville, 1997). However, the main challenge is now to enrich the framework for handling multivariate models; so, the scope of application will be enlarged to other appealing solutions like common components models. By using the univariate treatment of multivariate series as described in DK (2001, § 6.4), that could be done in a reasonable delay (2006).

6 CONCLUDING REMARKS

The libraries developed by the R&D cell of the statistical department of the National Bank of Belgium are an attempt to combine recent developments around the SSF and new technologies.
They are still in a prototyping phase; so, their algorithms and their results should be handled with some caution. Updates of the software and of the documentation will be regularly made available on a WEB site of the Bank (http://www.nbb.be/app/dqrd/index.htm).

Anyway, the present work already shows that the OO paradigm fits very well the versatility of the state-space models and that, through a good design, it is possible to achieve fast processing as well as flexibility. Reusability is also achieved by the use of standard technologies like Java and .NET.

As far as benchmarking is concerned, state-space techniques provide a coherent framework for the traditional methods. Thanks to their high performances, they make feasible a variety of analysis devices.

Finally, further improvements in the benchmarking domain should be obtained by means of a multivariate approach.

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Appendix 1. State-space representation of an ARIMA model.

The ARIMA process is defined by

$$\Delta(B)\Gamma(B)y(t) = \Theta(B)\varepsilon(t),$$

where

$$\Delta(\mathrm{B}) = 1 + \delta_1\mathrm{B} + \ldots + \delta_\mathrm{d}\mathrm{B}^\mathrm{d}$$
$$\Gamma(\mathrm{B}) = 1 + \varphi_1\mathrm{B} + \ldots + \varphi_\mathrm{p}\mathrm{B}^\mathrm{p}$$
$$\Theta(\mathrm{B}) = 1 + \theta_1\mathrm{B} + \ldots + \theta_\mathrm{q}\mathrm{B}^\mathrm{q}$$

are the differencing, auto-regressive and moving average polynomials. We also write:

$$\Phi(\mathrm{B}) = \Delta(\mathrm{B}) \bullet \Gamma(\mathrm{B}) = 1 + \phi_1\mathrm{B} + \ldots + \phi_\mathrm{p}\mathrm{B}^{\mathrm{p+d}}$$

Let ψ_i be the psi-weights of the Arima model, $\psi_{st,i}$ and $\gamma_{st,i}$, the psi-weights and the autocovariances of the differenced Arma model. We also define:

$$r = \max(p + d, q + 1)$$
$$s = r - 1$$

Using those notations, the state-space model can be written as follows[12]:

State vector:

$$\alpha(t) = \begin{pmatrix} y(t) \\ y(t+1|t) \\ \cdots \\ y(t+s|t) \end{pmatrix}$$

where $y(t+i|t)$ is the orthogonal projection of $y(t+i)$ on the subspace generated by $\{y(s) : s \le t\}$. Thus, it is the forecast function with respect to the semi-infinite sample.

System matrices:

Using the notations of § 3.1, the matrices of the model are

$$Z(t) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$h(t) = 0$$

$$T(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\phi_r & \cdots & \cdots & \cdots & -\phi_1 \end{pmatrix}$$

[12] See for example Gomez-Maravall (1994)

$$W(t) = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \vdots \\ \psi_s \end{pmatrix}$$

$$R(t) = I_r$$

$$Q(t) = \sigma^2$$

and the initial conditions can be written:

$$a_0 = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}$$
$$P_{*,0} = \Sigma V \Sigma'$$
$$P_{\infty,0} = \Lambda\Lambda'$$

V is the variance/covariance of the stationary model; it can be derived by the relationships:

$$V[i,0] = V[0,i] = \gamma_{st,i}$$
$$V[i,j] = V[i-1,j-1] - \psi_{st,i}\psi_{st,j}$$

Σ is a r x r lower triangular matrix with ones on the main diagonal; other cells are defined by the recursive relationship:

$$\Sigma[i,j] = -\delta_1\Sigma[i-1,j] - \ldots - \delta_d\Sigma[i-d,j]$$

with the convention $\Sigma[i,j] = 0$ if $i < 0$

Λ is a r x d matrix; its first d rows form an identity matrix; other cells are defined as above:

$$\Lambda[i,j] = -\delta_1\Lambda[i-1,j] - \ldots - \delta_d\Lambda[i-d,j]$$

Appendix 2. Definition of the IModel interface (using the C# syntax)

```
public interface IModel : IParametriseable
{
    // Observations of the model
    void SetY(double[] y);
    double Y(int t);


    // Information on the observations
    bool HasData{get;}
    // Number of available observations.
    int DataCount{get;}
    // Number of considered iterations. could be less than DataCount, if
    some observations are dropped or more than DataCount when forecasting
    is considered
    int EndPosition{get;set;}

    ///////////////
    // Matrices of the model at time t. See 3.1 for further explanations

    // Size of the state. Noted r in 3.1.
    int StateDim{get;}
    // utility of t
    bool IsTimeInvariant{get;}

    // Measurement matrix
    void Z(int t, double[]z);

    // Variance of the measurement disturbance term
    double H(int t);

    // Transition matrix
    void T(int t, Matrix tr);

    // Variance of the state disturbance terms
    void Q(int t, SymmetricMatrix q);
    void R(int t, int[] r);
    void W(int t, Matrix W);


    // Information on the matrices of the model
    bool HasMeasurementDisturbance{get;}
    bool HasR{get;}
    bool HasW{get;}

    // Length of R. Noted m in 3.1. 0<m<=r
    int ResCount{get;}
    // Size of Q. Noted e in 3.1. 0<e<=r
    int ResDim{get;}
    ///////////////

    // Initialisation

    // Initial state
    double[] A0{get;}
    // Variance of the initial non diffuse part
    SymmetricMatrix Pf0{get;}
    // Variance of the initial diffuse part
    SymmetricMatrix Pi0{get;}
```

```
    // Number of independent non-stationary components. equals rank of
    Pi0. Noted d in 3.1
    int NonStationaryDim{get;}
    // d > 0
    bool IsDiffuse{get;}
    ///////////////

    // Others informations

    bool IsValid{get;}

    // L = T - K*Z, K column-vector
    void L(int t, double[] K, Matrix l);

    ///////////////
    // Atomic operations

    // Forwards operations

    // xout = T*xin, xin column-vector
    void TX(int t, double[] xin, double[] xout);

    // Z*xin, xin column-vector
    double ZX(int t, double[] xin);

    // xout = Z*M, M matrix
    void ZM(int t, AbstractMatrix M, double[] xout);

    // V = T*V*T', the most time-consuming operation. It should be
    carefully optimized.
    void TVT(int t, SymmetricMatrix V);

    // Z*V*Z'
    double ZVZ(int t, SymmetricMatrix V);


    // Backwards operations

    // xout = xin*T, xin row-vector
    void XT(int t, double[] xin, double[] xout);

    // V = V + Z'*d*Z
    void VpZdZ(int t, SymmetricMatrix V, double d);

    // X = X + Z * d, X row-vector
    void XpZd(int t, double[] X, double d);

}
```

Appendix 3. Examples of code (C# syntax).

The goal of the following examples is just to give a fast overview of the capabilities of the library and of its conciseness. Real code needs of course much further documentation.

```
////////////////////////////////////////////////////////////////////////
// Example 1. Computation of the likelihood of a Chow-Lin/Litterman model
// following several ways. Quarterly series. Fixed parameter (-.9).
////////////////////////////////////////////////////////////////////////
// Construction of an (RW)AR1 ARIMA model
int freq=4;
SArimaSpecification spec=new SArimaSpecification(freq);
spec.P=1;
// spec.D=1; // for litterman

SArimaModel arima=new SArimaModel(spec);
arima.Parameters=new double[]{-.9};

// double[] yc=...
// double[][] x=..., xc=...
// We suppose that yc, x, xc are respectively the aggregated series
//(expanded with missing values), the related series and the cumulated
// related series.

// Contruction of a (Durbin-Koopman) SSF for the given ARIMA model
// (see 2.1.1 and appendix 1)
ArimaDKModel mSSF=new ArimaDKModel(arima);

// Construction of an optimized benchmarking model for the given ARIMA
// model (see 2.2.2)
ArimaExpander mxSSF=new ArimaExpander(arima, freq);
mxSSF.SetY(yc);

// Augmented model (see 2.1.4.2)
AugmentedModel agModel=new AugmentedModel();
agModel.Model= mxSSF;
agModel.Regressors=xc;
agModel.UseDiffuseRegressors(true);
double agLL= agModel.LogLikelihood;

// Extended model (see 2.1.4.1). Construction + Estimation
ExtendedModel exSSF=new ExtendedModel(mxSSF, xc);
ModelEstimation mEst=new ModelEstimation();
mEst.Model=exSSF;
double exLL=mEst.LogLikelihood;

// use of not optimized methods
// Generic expansion (see 2.2.3.1) followed by Extended model
GenericExpander gxSSF=new GenericExpander(mSSF,freq);
ExtendedModel exg=new ExtendedModel(gxSSF, xc);
exg.SetY(yc);
mEst.Model=exg;
double exgLL=mEst.LogLikelihood;

// Extended model followed by Generic expansion
ExtendedModel exm=new ExtendedModel(mSSF, x);
GenericExpander gex=new GenericExpander(exm, freq);
gex.SetY(yc);
mEst.Model=gex;
double gexLL=mEst.LogLikelihood;

// It is to be noted that the previous two solutions imply quite
```

```
// different SSF.
```

Following the length and the frequency of the series, the performances of the previous solutions differ sometimes substantially. The table below gives the number of function evaluations per second. (Context: C# implementation, Windows XP, 3 Ghz CPU, 512 MB RAM).

	Chow-Lin		**Litterman**	
	Quarterly series (15 years)	Monthly series (15 years)	Quarterly series (15 years)	Monthly series (15 years)
Augmented model	4500	2050	**3600**	**1750**
Extended model	**5600**	**2100**	3400	1550
Generic expansion + extended model	3900	1450	2100	1000
Extended model + generic expansion	3300	1150	1900	600

```
////////////////////////////////////////////////////////////////////////
// Example 2. Benchmarking for a local linear trend. No regressors.
// Estimation and smoothing.
////////////////////////////////////////////////////////////////////////

// Construction of the basic structural model
BSMSpecification bspec=new BSMSpecification();
bspec.HasSeas=false;
bspec.HasSlope=true;
BasicStructuralModel bsmSSF=new BasicStructuralModel(bspec, null, freq);
bsmSSF.IsNoiseInState=true;

// Generic expander.
GenericExpander bsmxSSF=new GenericExpander(bsmSSF,freq);
bsmxSSF.SetY(yc);

// Likelihood function (see 2.1.3)
ModelEstimation bsmEst=new ModelEstimation();
bsmEst.Model=bsmxSSF;

// Optimization through BFGS. BFGS needs an auxiliary object to check the
// validity of the parameters
BFGS bfgs=new BFGS();
BSMChecker checker=new BSMChecker();
checker.UseParamsTransformation=false;

if (bfgs.Minimize(bsmEst, checker)){
   bsmEst=(ModelEstimation) bfgs.Result;
   bsmxSSF=(GenericExpander)bsmEst.Model;
}

// Smoothing of the generic expander
SmoothingResults srslts=new SmoothingResults();
Smoother smoother=new Smoother();
smoother.Process(bsmxSSF, srslts);
// the results can be further used. For example, the following code
// yields the smoothed series (ys).
double[] zorig=new double[bsmxSSF.StateDim];
double[] ys=new double[yc.Length];
for (int i=0; i<ys.Length; ++i){
   bsmxSSF.ZOriginal(i, zorig);
   ys[i]=srslts.ZComponent(zorig);
}
```